Nonlinear Differential Equations

Math 102 Section 102
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Annoucements

- OSH remarking requests
- Midterm solution

Last time: exponential decay

Q1. The amount of ^{14}C in a sample decreases at a rate proportional to how much ^{14}C is present. Write down a DE governing the amount of ^{14}C at time t.

A. $C'(t) = kC(t)$, $k > 0$ solution grows

B. $C'(t) = -kC(t)$, $k > 0$

C. $C(t) = C_0 e^{kt}$ not a DE

D. $C'(t) = C_0 e^{-kt}$ rate not proportional to amount

Last time: exponential decay

What's the family of general solutions to the following DE?

$$\frac{dx}{dt} = -kx.$$

$$x(t) = Ce^{-kt}, \quad \text{for some } C.$$

Remark:

- Any function in the form of $x(t) = Ce^{-kt}$ is a solution to the DE. (we can prove easily)
- Any solution to the DE must be in the form of $x(t) = Ce^{-kt}$. (temporarily beyond our reach)

Last time: summary

- Definitions related to differential equations.
- Initial conditions specify a unique solution.
- Radioactive decay and population growth are described by a simple DE.

Today: learning goals

- Explain the concepts of linear and nonlinear DEs
- Derive DEs and scale them (moved to next Monday)

Derive a simple DE

Q2. The energy (calories) carried by a hummingbird is $F(t)$. Suppose that the intake takes place at a constant rate a and energy consumption is proportional to the amount of energy being carried with constant b. Which differential equation describes how $F(t)$ changes?

A. $\frac{dF}{dt} = a - bt$

B. $\frac{dF}{dt} = (a - b)t$

C. $\frac{dF}{dt} = (a - b)F$

D. $\frac{dF}{dt} = a - bF$

Rate of change of energy carried = rate of addition - rate of consumption $= a - b \cdot F$.

“Easy” and “hard” differential equations

- (Uncontrolled) Population growth

$$\frac{dN}{dt} = kN, \quad k > 0$$

- Exponential decay:

$$\frac{dC}{dt} = -kC, \quad k > 0$$

- Hummingbird metabolism

$$\frac{dF}{dt} = a - bF$$

- Prey population growth under predation

$$\frac{dx}{dt} = rx - \frac{Kx}{a+x} \text{ or } \frac{dx}{dt} = rx - \frac{Kx^2}{a^2+x^2}$$

Linear and nonlinear DEs

Definition (Linear DE)

A differential equation is called a linear (first order) DE if it can be written as

$$\alpha \frac{dy}{dt} + \beta y + \gamma = 0$$

for some constants α, β, γ that do not depend on y or y' but may depend on t.

- The key idea is, terms containing the function y or $\frac{dy}{dt}$ are linear in y and $\frac{dy}{dt}$.
- Correspondingly, nonlinear (first order) DEs have a nonlinear term in y or $\frac{dy}{dt}$. Examples: $y\frac{dy}{dt} = -1$ or $\frac{dy}{dt} = y^2$
- First order: only involving the first derivative of the unknown function

Linear and nonlinear DEs

Q3. Which of the following differential equations is nonlinear?

A. $\frac{dy}{dt} = t^2$

B. $\frac{dy}{dt} = \ell t - ky$

C. $\frac{dy}{dt} = \sin(t)$

D. $\frac{dy}{dt} = \sin(y)$

The equations in A,B, and C, are linear in y and $\frac{dy}{dt}$

Linear and nonlinear DEs

Put the following DEs in the chart below:

$$\frac{dy}{dx} = y \qquad 2\frac{dy}{dx} = y - \frac{dy}{dx} \qquad y\frac{d^2y}{dx^2} = 1 \qquad \frac{d^2y}{dx^2} = 2 \qquad \frac{dy}{dx} = xy$$

	first order	higher order
linear		
nonlinear		

Linear and nonlinear DEs

Put the following DEs in the chart below:

	first order	higher order
linear	$\frac{dy}{dx} = y$ $2\frac{dy}{dx} = y - \frac{dy}{dx}$ $\frac{dy}{dx} = xy$	$\frac{d^2y}{dx^2} = 2$
nonlinear		$y\frac{d^2y}{dx^2} = 1$

“Hard” differential equations

Why nonlinear equations?

- Because many phenomena are nonlinear!

How can we study them?

- Next week: qualitative methods (geometrical theory)
- The following week: analytical and numerical methods

Answers

1. B
2. D
3. D