Differential Equations for Growth and Decay

Math 102 Section 102, Mingfeng Qiu

Today: learning goals

1. Apply logarithms to solve application problems (moved from Monday)
2. Explain what are differential equations and initial conditions
3. Derive the differential equation describing exponential growth or decay
4. Explain doubling time and half-life

Applications of logarithms

1. Genetic Diversity (worksheet)
2. Log-log plots and semi-log plots

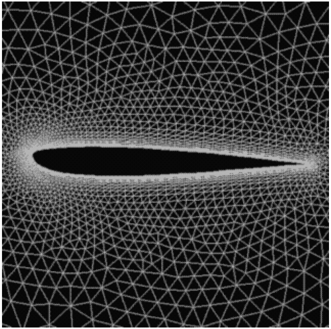
Log plots

- A log-log plot is a plot on which you plot log(y) versus log(x) (instead of y versus x) (usually used for power functions)
- A semi-log plot is a plot where you plot log(y) versus x (instead of y versus x) (usually used for exponential functions)

Log plots

- In scientific computing, we use the computer to solve an equation numerically, i.e., obtaining an approximation. This approximation is obtained using a mesh.

Wikipedia



Zhukovskii airfoil by NASA

Log plots

Let

- exact solution $= u_e$
- mesh size $= h$
- numerical solution on the mesh with size $h = u_h$

The accuracy of the approximation depends on the mesh size h.
As a toy model, we

$$|u_h - u_e| = Ch^n$$

- What is n?

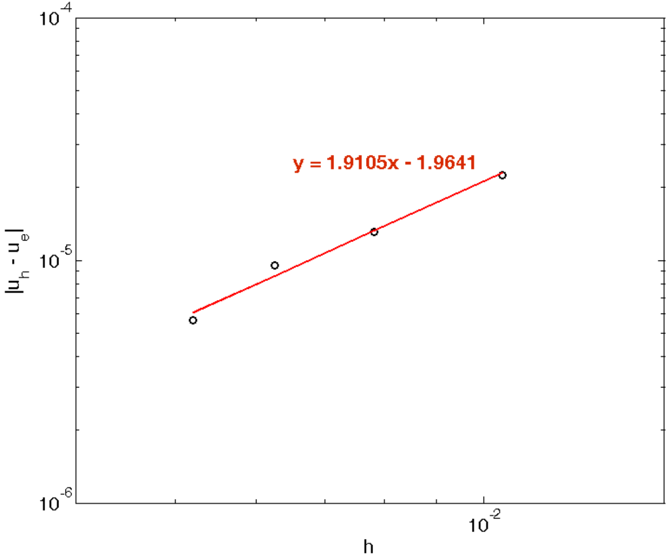
Log plots

- Take logarithm:

$$\ln(|u_h - u_e|) = n \ln h + \ln C$$

- Perform computations on a series of meshes, with different sizes h
- Linear fit - least squares

Log plots



Summary for logarithms

- $\ln(x)$ is the inverse function for e^x
- $\frac{d}{dx} \ln x = \frac{1}{x}$
- $\frac{d}{dx} a^x = a^x \ln a$
- Logarithms can be used to solve exponential equations (e.g. bacteria)
- Log-log plots and semi-log plots are ubiquitous in science.
- Information theory is cool! Check it out https://en.wikipedia.org/wiki/Information_theory

Differential equations for growth and decay

Population growth

- $x(t) =$ number of individuals at time t
- The size of the population satisfies the differential equation:

$$\frac{dx}{dt} = rx$$

- (assuming uncontrolled growth, for now)
- Units?
  - $\frac{dx}{dt}$ has units of $\frac{1}{T}$(number per time)
  - x has unit of 1 (number)
  - r has unit of $\frac{1}{T}$ (r is the per capita growth rate)

Population growth

- The size of the population satisfies the differential equation:

$$\frac{dx}{dt} = rx$$

- To solve this equation means finding a function $x(t)$ which satisifies the equation. A solution to this differential equation is

$$x(t) = 10e^{rt}$$

since

$$\frac{dx}{dt} = r(10e^{rt}) = rx$$

How many solutions does a differential equation have?

Q1. Which of the following functions satisfy the differential equation

$$\frac{dy}{dt} = ky.$$

A. $y = e^{kt}$

B. $y = 10e^{kt}$

C. $y = 2e^{kt}$

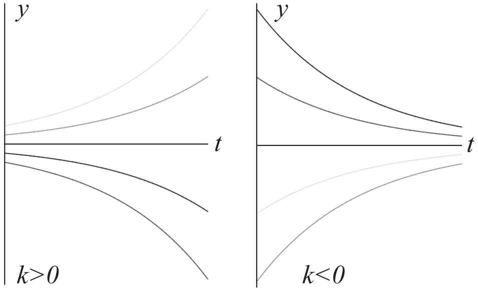
D. $y = ke^{kt}$

E. All of the above

Remark: it turns out that all solutions to this equation must have the same form $y(t) = Ce^{kt}$. They are called general solutions.

How many solutions does a differential equation have?

- The function $y = Ce^{kt}$, for any constant, C is a solution to the differential equation.



- For $k > 0$, the solutions grow exponentially
- For $k < 0$, the solutions decay exponentially

What determines *the* solution?

- To determine *the* solution, there must be additional information.
- The initial condition sets the value of C.
- If $\frac{dy}{dt} = ky$, with the initial condition $y(0) = y_0$, then

  $$y(t) = Ce^{kt} \Rightarrow y(0) = Ce^0 = C$$

  so $C = y_0$.
- The solution is

  $$y(t) = y_0 e^{kt}.$$

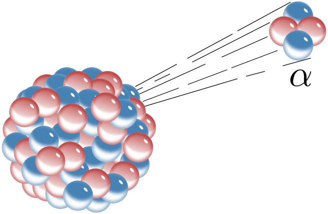
- It's called a specific solution.

Definitions

Definition (Differential equations)

- A differential equation is a mathematical equation that relates one or more derivatives of some function to the function itself.
- A solution to a differential equation is a function that satisfies that equation.
- An initial condition specifies a known value at some specific time point (usually at time $t = 0$).
- Among the family of general solutions, one can determine a specific solution by using an initial condition.

Radioactive decay



- Over 5-second period, the probability of a radioactive atom decays is $\frac{1}{100}$.
- The behavior of a single atom is unpredictable.
- But we can predict the behavior accurately with a large number of them.
  - What if we have 1000 atoms?

Radioactive decay

- The probability of an atom decaying per unit time is k. That means in a small time interval h, the probability of an atom decaying is kh.
- Let $Q(t)$ be the number of the radioactive atoms (isotope) at time t.

- Derive a differential equation describing the evolution of $Q(t)$ in time.
- At what time $t_{1/2}$ is $Q(t_{1/2}) = \frac{1}{2}Q_0$?

— Document camera

Carbon dating (worksheet)

Researchers at Charlie Lake in BC have found some artifacts. For instance, a butchered bison bone that contains 0.25 mg of ^{14}C isotope. A comparable bone of a bison alive today contains about 1 mg of ^{14}C. We know that the half-life of ^{14}C is 5730 years. How many years ago could human habitation be dated back to in this region[1]?

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

Summary

- Definitions related to differential equations
- Initial conditions specify a unique solution
- Radioactive decay and population growth are described by a simple DE

Answers

1. E

Related Exam Problems

Question 7: [12 points] Solutions to the differential equation

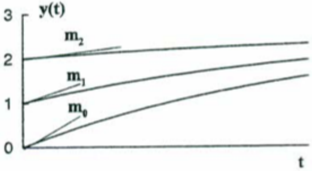
$$\frac{dy}{dt} = a - by$$

starting at three different initial values are shown in the graph. Also shown are the tangent lines to these curves at $t = 0$. You are given the following information about the slopes of these tangent lines:

(i) The slope m_0 is five times the slope m_2.

(ii) The slope m_1 is 3.

Use this information to answer the following questions:



(a) Determine the values of the constants a and b. (Justify your answer.)

(b) Determine the value that $y(t)$ will approach after a long time on any of these curves.

(c) Find the value of $y(t)$ at time $t = 0.2$ given that $y(0) = 1$. Round off your answer to 2 decimal places and justify how this answer was obtained.