Least Squares and Chain Rule

Math 102 Section 102
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Oct. 19, 2018

Annoucements

- I'm not here for the rest of today, neither can I answer emails.
  - Emergency for homework: Dr. Lisanne Rens @MATX 1118, 2-3 pm
- How to study for midterm
  - Exercise: "exam related problems", questions at the back of each chapter in the open textbook, practice problems and from past exams (link found on Canvas Midterm Information page).
  - Review your past WeBWork and OSH assignments, in particular the ones you didn't get right.
- Q&A Sessions
  - Mon, Oct 22, 3-7 pm @ BUCH A102 (I will be there 5-6 pm)
  - Tue, Oct 23, 3-7 pm @ CHBE 101

We're on the same boat...

- You are definitely welcome to come to me and ask questions.
- My office hour is running at (more than) full capacity... To accommodate everyone, please consider using MLC/Piazza for shorter questions.

- Office Hours: Tue 2:30-4 pm; Wed 3-4 pm
- Email: mqiu@math.ubc.ca
- Webpage: http://www.math.ubc.ca/~mqiu/m102.html

Agenda

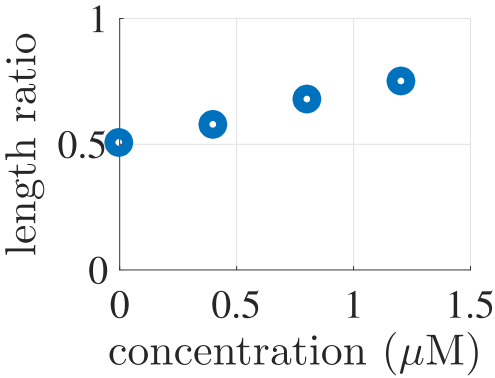
Last time:

- Optimal foraging and marginal value theorem
- Measuring central tendency: average of a data set
- Fitting data using a line without intercept $y = ax$

Today:

- Fitting data using a line with intercept $y = ax + b$
- Revisiting chain rule

Fitting a line to data



Fitting a line without intercept

Q1. Suppose you have data (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and wish to fit a line $y = ax$ through these points.
A residual for (x_i, y_i) is

A. $r_i = y_i^2 + x_i^2$

B. $r_i = y_i - ax_i$

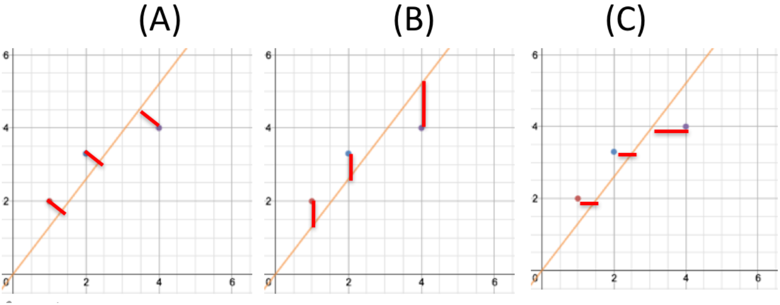
C. $r_i = a^2(y_i^2 + x_i^2)$

D. $r_i = y_i - x_i$

E. $r_i = x_i - y_i$

Fitting a line without intercept

Q2. Suppose you have data (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and wish to fit a line $y = ax$ through these points.
Which graph below shows the residuals, $(y_i - ax_i)$?



Graph B

Fitting a line without intercept

Q3. To get the best fit line $y = ax$ in the least squares sense, we should minimize

A. $f(a) = \sum_{i=1}^{4}(x_i - y_i)^2$

B. $f(a) = \sum_{i=1}^{4}(y_i - ax_i)^2$

C. $f(a) = \sum_{i=1}^{4}|y_i - ax_i|$

D. $f(a) = \sum_{i=1}^{4}(y_i - ax_i)$

Minimize the sum of square residuals.

Fitting a line without intercept

To find the line, $y = ax$, of best fit to n data points (x_i, y_i), minimize the sum of square residuals:

$$f(a) = \sum_{i=1}^{n} (y_i - ax_i)^2.$$

Note that

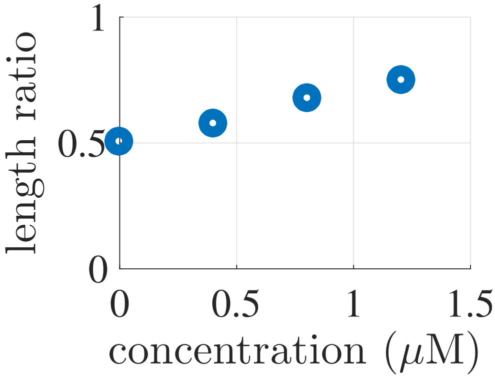
$$f'(a) = -2 \left(\sum_{i=1}^{n} y_i x_i - a \sum_{i=1}^{n} x_i^2 \right) = 0$$

so

$$a = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}$$

Fitting a line with intercept

SU5402 concentration μM	0	0.4	0.8	1.2
lef1/PLLP length ratio	0.51	0.58	0.68	0.75



Fitting a line with intercept

- Goal: use a general line $y = ax + b$ to approximate the data set.
- Idea: least squares
- Residual at one data point:

$$r_i = y_i - ax_i - b$$

- Minimize the SSR:

$$\sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$

Fitting a line with intercept

Some intermediate steps:

- Average x and y values

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

- Average product of x and y values

$$P_{\mathsf{avg}} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$$

- Average of x values squared

$$X^2_{\mathsf{avg}} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

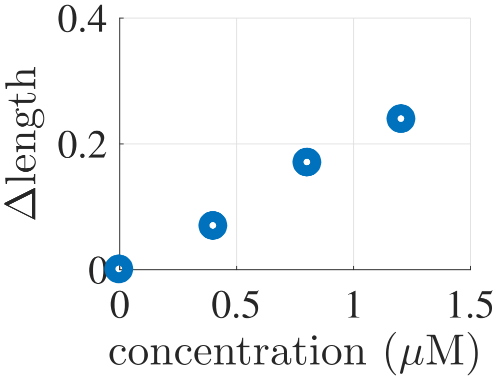
Fitting line:

$$a = \frac{P_{\mathsf{avg}} - \bar{x}\bar{y}}{X^2_{\mathsf{avg}} - \bar{x}^2}, \qquad b = \bar{y} - a\bar{x}$$

Spreadsheet example

Link

Fitting a line to data: summary



- A statistical model is a function that approximates a set of data.
- Idea: minimize a certain measure of the difference between the data and approximations.
- Example: least squares with average, $y = ax$, or $y = ax + b$.

Revisiting the chain rule

Recall: composite functions

$$k(u) = k(u(o))$$

Recall: chain rule

If $y = f(u)$ and $u = g(x)$ are both differentiable functions and $y = f(g(x))$ is the composite function, then the chain rule of differentiation states that

$$y' = f'(u)g'(x) = f'(g(x))g'(x)$$

or written in another way

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example

$$\frac{d}{dx}\sqrt{x^5+22} =$$

Let

$$y = f(u) = \sqrt{u}, \qquad u(x) = x^5 + 22.$$

$$\begin{aligned}
\frac{dy}{dx} &= f'(u(x))u'(x) \\
&= \frac{1}{2}(u(x))^{-\frac{1}{2}} \cdot u'(x) \\
&= \frac{1}{2}(x^5+22)^{-\frac{1}{2}} \cdot 5x^4 \\
&= \frac{5x^4}{2\sqrt{x^5+22}}
\end{aligned}$$

Practice problems

- $\frac{d}{dx}(x^2 + 17x - 9)^5$
- $\frac{d}{dx}\sqrt{(2x^5 + x^3)^4 + 22x^2}$
- A sphere's volume is increasing at a rate of 3m^3 per minute. How fast is its radius increasing when its radius is 1 metre?
- Chapter 8 of the online textbook: Examples 8.11, 8.13

(Chapter 8 of the online textbook is a good place to look for optimization problems involving the chain rule.)

Chain rule in optimization

- Chapter 8 of the Course Notes is a good place to look for optimization problems involving the chain rule.
- These problems can be quite tricky.

Answers

1. B
2. B
3. B