Worksheet: optimization I
Math 102 Section 102
Oct. 12, 2018

1. Complete the steps in this optimization problem.

(i) Can you write down our objective function?

(ii) Can you write down our constraint?

(iii) We want to eliminate one variable from the objective function using our constraint. Which variable should we eliminate? Why?

(iv) Carry out the elimination and write the objective function in terms of only one independent variable.

(v) Use calculus to identify extrema.

(vi) What does the extremum tell us? Explain in words how this helps Kepler select the wine barrel.

2. The sum of two positive numbers is 20. Find the numbers if

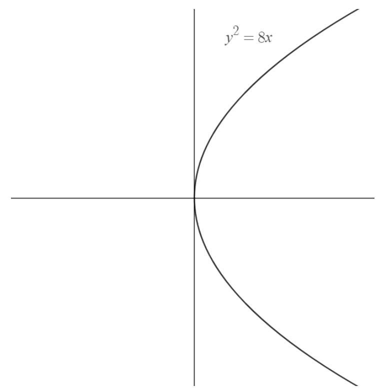
(i) their product is a maximum.

(ii) the sum of their squres is a minimum.

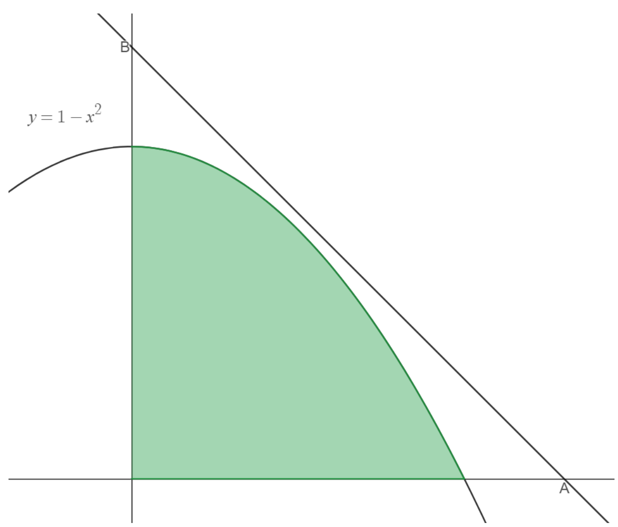
(iii) the product of the square of one and the cube of the other is at a maximum

3. You have L metres of rope, and you want to use it to form a circle and a square. How would you enclose the most area? The least?

4. Find the minimum distance from the point $(a, 0)$ to the parabola $y^2 = 8x$.



5. Find a point A on the positive x-axis and a point B on the positive y-axis such that (i) the triangle AOB contains the first quadrant portion of the parabola $y = 1 - x^2$ and (ii) the area of the triangle AOB is as small as possible



Worksheet solution (Oct 12, 2018) (Friday)

2. Suppose one number is x, then the other is $20-x$. The domain is $0<x<20$, which guarantees both numbers are positive.

(i) Maximize $f(x) = x(20-x) = 20x - x^2$

$f'(x) = 20-2x \Rightarrow$ CP: $x=10$

$f''(x) = -2 < 0 \Rightarrow$ By SDT, local max at $x=10$

$f'(x) > 0$ when $0<x<10$, $f(x)$ increases.

$f'(x) < 0$ when $10<x<20$, $f(x)$ decreases.

Hence the local max is the global max.

Thus $x=10$, $20-x=10$.

(ii) Minimize $f(x) = x^2 + (20-x)^2 = 2x^2 - 40x + 400$

With the same procedure as above, the absolute min is attained at $x=10$, $20-x=10$

(iii) Maximize $x^3(20-x)^2 = f(x)$

(Note: one can also maximize another function $x^2(20-x)^3$, which will give the same result, just with the order of the two numbers switched. Choosing $x^2(20-x)^3$ is a bit harder, since you need to possibly expand $(20-x)^3$.)

$f(x) = x^5 - 40x^4 + 400x^3$

$f'(x) = 5x^4 - 160x^3 + 1200x^2 = 5x^2(x-12)(x-20)$

$\Rightarrow$ CP: $x = 0, 12, 20$

x		0		12		20	
$f'(x)$	$+$		$+$		$-$		$+$
$f(x)$	$\nearrow$		$\nearrow$		$\searrow$		$\nearrow$

model domain

So $f(x)$ attains global max at $x = 12$, $20 - x = 8$.

3. Let r be the radius of the circle. Then

perimeter of the circle $= 2\pi r$

perimeter of the square $= L - 2\pi r$

edge length of the square $= \frac{L}{4} - \frac{\pi r}{2}$

total area = area of the circle + area of square.

$$= \pi r^2 + \left(\frac{L}{4} - \frac{\pi}{2} r\right)^2$$

$$= \left(\frac{\pi^2}{4} + \pi\right) r^2 - \frac{\pi L}{4} r + \frac{L^2}{16}$$

The domain is $0 \le r \le \frac{L}{2\pi}$

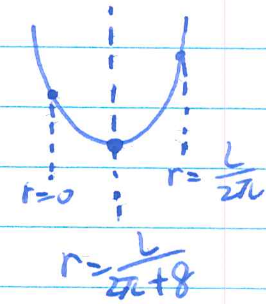
all used for square

all used for circle

If you go through the CP - extrema steps, you will find that the only local min is at $r = \frac{L}{2\pi+8}$ (also is global min)

$f(x)$ decreases when $r < \frac{L}{2\pi+8}$

$f(x)$ increases when $r > \frac{L}{2\pi+8}$



Therefore, the global max has to be one of the end points of the feasible domain $[0, \frac{L}{2\pi}]$

$r = 0 \Rightarrow$ total area $= \frac{L^2}{16}$

$r = \frac{L}{2\pi} \Rightarrow$ total area $= \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi} > \frac{L^2}{16}$ since $\pi < 4$.

So you always want to just enclose a circle with all you rope for the <u>most area</u>. i.e. $r = \frac{L}{2\pi}$.

To enclose the <u>least area</u>, we choose the global min $r = \frac{L}{2\pi+8}$.

4. Let $d(x,y)$ be the distance from $(a,0)$ to a point (x,y).

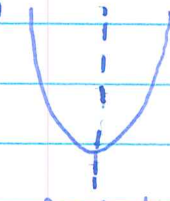
① If $a \le 0$, apparently the closest point on $y^2 = 8x$ to the point $(a,0)$ is the left-most point on the parabola, i.e. $(0,0)$. Then $d(0,0) = -a$.

② If $a>0$, $d^2=(a-x)^2+y^2$ for (x,y) on the parabola

$$= x^2-2ax+a^2+\underline{\underline{8x}}$$

$$= x^2+(8-2a)x+a^2$$

for the function $f(x)=x^2+(8-2a)x+a^2$, the global min is at $x=a-4$



But notice for the parabola, $0\le x<\infty$

If $0<a\le 4$, the global min on the feasible domain is at $x=0$. $d(0,0)=a$

If $a\ge 4$, the global min is achieved at $x=a-4$.

$$d(a-4,\pm\sqrt{8a-32})=\sqrt{16+8a-32}=2\sqrt{2a-8}$$

5. The conditions (i) and (ii) imply that AB has to be a tangent line to the curve $y=1-x^2$ in the first quadrant.

Assume that AB is the tangent line at the point $(a,1-a^2)$ on the curve.

It has to be that $0<a\le 1$ x-intercept of $y=1-x^2$ on the $+x$-axis.

$f'(x)=-2x$.

So the tangent line equation is

$$y-(1-a^2) = -2a(x-a)$$

$$y = -2ax + a^2+1$$

point A

This line intercepts the x-axis at $\left(\frac{a}{2}+\frac{1}{2a}, 0\right)$.

and intercepts the y-axis at $(0, a^2+1)$ point B.

Then the area of the triangle AOB is

$$\frac{1}{2}\cdot OA\cdot OB = \frac{1}{2}(a^2+1)\left(\frac{a}{2}+\frac{1}{2a}\right)$$

This can be viewed as a function of a:

$$S(a) = \frac{1}{2}(a^2+1)\left(\frac{a}{2}+\frac{1}{2a}\right)$$

$$= \frac{1}{4}\left(a^3+2a+\frac{1}{a}\right).$$

and we know $0<a\leq 1$

$$S'(a) = \frac{1}{4}\left(3a^2+2-\frac{1}{a^2}\right) = \frac{1}{4a^2}(3a^4+2a^2-1) = \frac{1}{4a^2}(3a^2-1)(a^2+1)$$

$S'(a)=0 \Rightarrow$ CP: $a=\pm\frac{1}{\sqrt{3}}$. Reject $a=-\frac{1}{\sqrt{3}}$ since it's outside our domain.

When $0<a<\frac{1}{\sqrt{3}}$, $S'(a)<0 \Rightarrow S(a)$ decreases.

When $\frac{1}{\sqrt{3}}<a\leq 1$, $S'(a)>0 \Rightarrow S(a)$ increases.

Thus on the interval $0<a\leq 1$, $S(a)$ attains global min at $a=\frac{1}{\sqrt{3}}$. This gives:

point A: $\left(\frac{2\sqrt{3}}{3}, 0\right)$ point B: $\left(0, \frac{4}{3}\right)$.