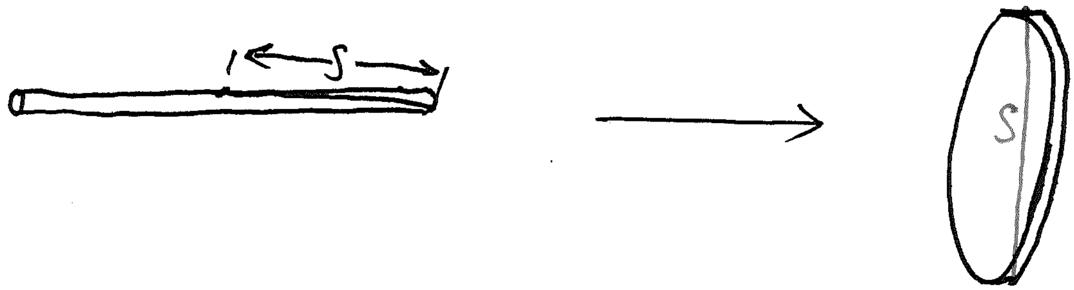
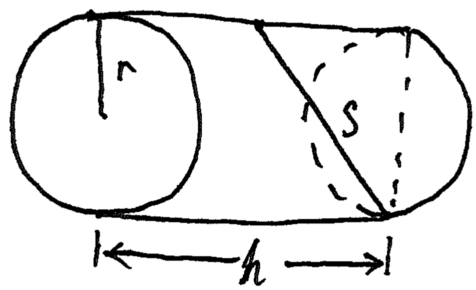


amount of wine: 0 ↗ + ↘ 0

asymptotic reasoning:

There must be a shape of cylinder that gives the most wine.



Stick wet length: S

radius: r

height: h

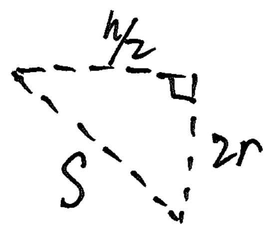
volume: V

maximize V when S is a constant.

objective function:

$$V = \pi r^2 h$$

constraint:



$$\left(\frac{h}{2}\right)^2 + (2r)^2 = S^2$$

$$\Rightarrow \quad 4r^2 = S^2 - \frac{h^2}{4}$$

$$r^2 = \frac{S^2}{4} - \frac{h^2}{16}$$

$$\Rightarrow V = \pi\left(\frac{S^2}{4} - \frac{h^2}{16}\right)h = \frac{\pi}{16}\left(4S^2h - h^3\right)$$

$$\frac{dV}{dh} = \frac{\pi}{16}\left(4S^2 - 3h^2\right)$$

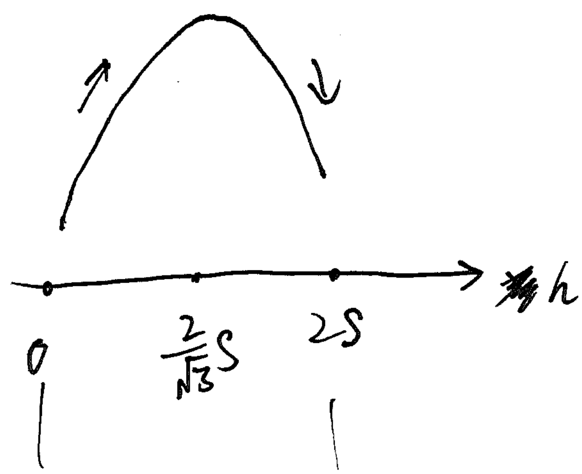
$\frac{dV}{dh} = 0 \Rightarrow 3h^2 = 4S^2 \qquad h = \pm \frac{2}{\sqrt{3}} S$

Only ~~need~~ $h = \frac{2}{\sqrt{3}} S > 0$ is in the model domain

When $h < \frac{2}{\sqrt{3}} S$, $\quad 3h^2 < 4S^2 \Rightarrow \frac{dV}{dh} > 0 \quad \nearrow$

When $h > \frac{2}{\sqrt{3}} S$, $\quad 3h^2 > 4S^2 \Rightarrow \frac{dV}{dh} < 0 \quad \searrow$

By FDT $\Rightarrow$ V attains local max at $h = \frac{2}{\sqrt{3}} S$.



So the local max is the global max on $[0, 2S]$.

The above analysis tells us that to gain the most wine with the same price, or the same length of the wet part S, Kepler should choose a wine barrel with height $h = \frac{2}{\sqrt{3}} S$. For this barrel, the radius is given by:

$$r^2 = \frac{S^2}{4} - \frac{h^2}{16} = \frac{1}{6} S^2$$

$$r = \frac{S}{\sqrt{6}}$$