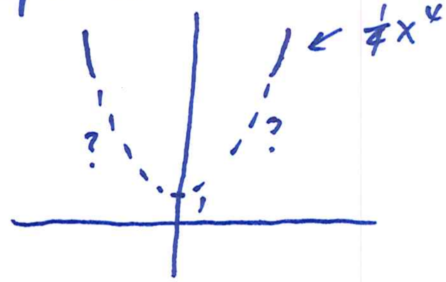
Sketch $f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$

⓪ asymptotics:

$|x|$ close to 0: $f(x) \approx 1$

$|x| \to \infty$: $f(x) \approx \frac{1}{4}x^4$

rough sketch:



① zeros:

Set $f(x)=0 \Rightarrow \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 = 0$

$x^4 - 8x^3 + 18x^2 + 4 = 0$

$x^2(x^2 - 8x + 18) + 4 = 0$

Consider $y = x^2 - 8x + 18 \longrightarrow$ quadratic.

Discriminant $\Delta = (-8)^2 - 4 \cdot 18 = 64 - 4 \cdot 18 = 4 \cdot 16 - 4 \cdot 18 = -8 < 0$

$\Rightarrow x^2 - 8x + 18 > 0, \ \forall x$

$\Rightarrow x^2(x^2 - 8x + 18) + 4 > 0, \ \forall x$

$\Rightarrow$ no zeros!

② CPs:

$f'(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2$. Set $f'(x) = 0$

$\Rightarrow x = 0, 3$ —— CPs

③ potential IPs:

$f''(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$ Set $f''(x) = 0$

$\Rightarrow x = 1, 3$ —— possible IPs

④ classify

x	$x<0$	0	$0<x<1$	1	$1<x<3$	3	$x>3$
f'	$-$		$+$		$+$		$+$
f	↘		↗		↗		↗
f''	$+$		$+$		$-$		$+$
f	∪		∪		∩		∪

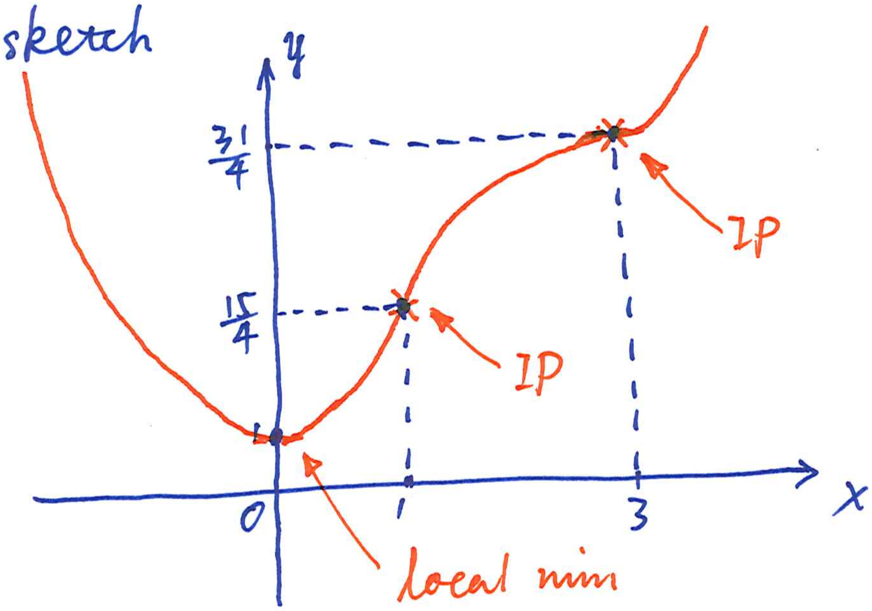
FDT: $f(x)$ has a local min at $x=0$, $f(0)=1$

$f(x)$ has IPs at $x=1, 3$.

$f(1) = \frac{15}{4}$

$f(3) = \frac{31}{4}$

⑤ sketch



Worksheet: Sketching functions using calculus tools
Math 102 Section 102
Oct. 3, 2018

Sketch the following function (You may need more than one piece of paper)

$$f(x) = \frac{(x-1)^2}{x^3}$$

Tips:

- Step 0: asymptotics and *discontinuities*
- Step 1: identify zeros
- Step 2: first derivative: identify CPs
- Step 3: second derivative: identify potential IPs
- Step 4: make a table: classify all the special points and characterize the shape of the function
- Step 5: sketch

Sketch $\frac{(x-1)^2}{x^3}$

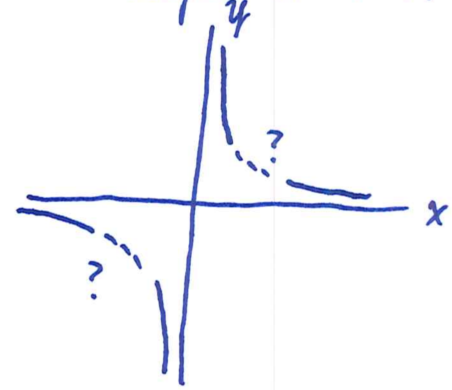
⓪ asymptotics :

$x \to 0-$: $f(x) \to -\infty$
$x \to 0+$: $f(x) \to +\infty$ } blow-up discontinuity

$x \to -\infty$: $f(x) \to 0-$

$x \to +\infty$: $f(x) \to 0+$

rough sketch:



① zeros :

Set $f(x) = 0 \Rightarrow \frac{(x-1)^2}{x^3} = 0 \Rightarrow x = 1$ — zero

② CPs :

$$f'(x) = \frac{2(x-1)\cdot x^3 - (x-1)^2 \cdot 3x^2}{x^6} = \frac{2x(x-1) - 3(x-1)^2}{x^4}$$

$$= \frac{(x-1)(2x-3x+3)}{x^4} = -\frac{(x-1)(x-3)}{x^4}$$

Set $f'(x) = 0 \Rightarrow -\frac{(x-1)(x-3)}{x^4} = 0 \Rightarrow x = 1, 3$

Hence, $x = 0, 1, 3$ are CPs.

③ potential IPs :

$$f'(x) = -\frac{(x-1)(x-3)}{x^4} = -\frac{x^2-4x+3}{x^4}$$

$$f''(x) = -\frac{(2x-4)\cdot x^4 - (x^2-4x+3)\cdot 4x^3}{x^8}$$

$$= -\frac{2x^2-4x-4(x^2-4x+3)}{x^5}$$

$$f''(x) = -\frac{-2x^2+12x-12}{x^5} = \frac{2}{x^5}(x^2-6x+6)$$

$$= \frac{2}{x^5}(x-3+\sqrt{3})(x-3-\sqrt{3})$$

Set $f''(x)=0 \Rightarrow \frac{2}{x^5}(x-3+\sqrt{3})(x-3-\sqrt{3})=0 \Rightarrow x = 3\pm\sqrt{3}$

Hence, $x=0,\ 3\pm\sqrt{3}$ are possible IPs.

④ classify

x		0		1		$3-\sqrt{3}$		3		$3+\sqrt{3}$	
f'	$\ominus$		$\ominus$		$\oplus$		$\oplus$		$\ominus$		$\ominus$
f	$\searrow$		$\searrow$		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$
f''	$\ominus$		$\oplus$		$\oplus$		$\ominus$		$\ominus$		$\oplus$
f	$\frown$		$\smile$		$\smile$		$\frown$		$\frown$		$\smile$

FDT: $f(x)$ has
local min @ $x=1$
local max @ $x=3$
$f(1)=0 \quad f(3)=\frac{4}{27}$

$f(x)$ has IPs @ $x=0,\ 3\pm\sqrt{3}$

$f(x)$ blows up as $x\to 0$

$f(3-\sqrt{3}) \approx 0.0352$

$f(3+\sqrt{3}) \approx 0.1314$

⑤ sketch