What do derivatives tell us about functions?

Math 102 Section 102
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Due due due due due...

- Oct 1 (Today): Pre-lecture 5.1
- Oct 3 (Wednesday): Pre-lecture 5.2
- Oct 4 (Thursday): Assignment 4

Assignments due: 9:00 pm

Announcements

- Spreadsheet resources (link)
- Midterm: Thursday, Oct 25
- Midterm signup on Canvas: choose one from three time slots
  - 5:00-6:30
  - 6:00-7:30
  - 7:00-8:30

This week

Sketching functions more accurately using calculus

- Characterize important points and shape features of functions
  - A lot of definitions today!
- Practice

Newton's method

Q1. We wish to find a good approximation to $7^{\frac{2}{3}}$ using Newton's Method. What function $f(x)$ is the best to help us compute this value?

A. $f(x) = x^{\frac{2}{3}} - 7$

B. $f(x) = x^2 - 343$

C. $f(x) = x^{\frac{3}{2}} - 7$

D. $f(x) = x^3 - 49$

$$x = 7^{\frac{2}{3}} \iff x^3 - 49 = 0$$

Newton's method

Q2. We wish to find a good approximation to $7^{\frac{2}{3}}$ using Newton's Method and $f(x) = x^3 - 49$. What value of x_0 would you chose as your starting estimate if you were planning to do the calculations by hand?

A. $x_0 = 4$

B. $x_0 = 7$

C. $x_0 = 8$

D. $x_0 = \sqrt{512}$

$$x_0 = 8^{\frac{2}{3}} = 4$$

What do derivatives tell us about a function?

Increasing vs. decreasing

Definition (Monotonic function)

- A function $f(x)$ is increasing on some interval I if

$$f(a) < f(b), \quad \forall a, b \text{ in } I \text{ and } a < b$$

- A function $f(x)$ is decreasing on some interval I if

$$f(a) > f(b), \quad \forall a, b \text{ in } I \text{ and } a < b$$

- Note: no reference to $f'(x)$!

What about $f'(x)$?

If $f(x)$ is differentiable on some interval...

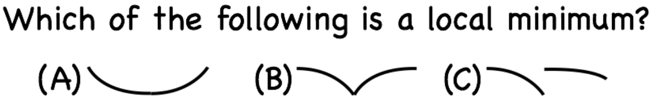
- If $f'(x) > 0$ on some interval,then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on some interval, then $f(x)$ is decreasing on that interval.
- If $f'(x) = 0$ on some interval, then $f(x)$ is constant on that interval.
- Want justification for these facts? Look up the Mean Value Theorem.

Local extrema (minima or maxima)

Definition (Local extrema)

- A point a is a local minimum of a function $f(x)$ provided that $f(x) > f(a),\ \forall x \neq a$ on an interval around a.
- A point a is a local maximum of a function $f(x)$ provided that $f(x) < f(a),\ \forall x \neq a$ on an interval around a.

Q3.



Critical points

- Local extrema belong to a wider class of points called critical points.

Definition (Critical point)

A critical point (CP) of $f(x)$ is a point a at which $f'(a) = 0$ or $f'(a)$ is not defined even though $f(a)$ is defined. (slightly different from Definition 6.2 in your textbook)

- A local extremum is necessarily a CP, but not the other way around.

Critical points

Q4. If $f'(x)$ goes from $-$ to 0 to $+$, then $x = a$ is a maximum of $f(x)$.

A. True

B. False

Critical points

Q5. If $f'(x)$ changes sign at a CP, then the CP is an extremum (min/max) of $f(x)$.

A. True

B. False

(Q5) This is called the first derivative test.

Critical points

Q6. If $f'(a) = 0$ and $f''(x) > 0$, then $x = a$ is a minimum of $f(x)$.

A. True
B. False

- $f'(x)$ changes from $-$ to 0 to $+$.

Critical points

Q7. If $f'(a) = 0$ and $f'(x)$ is differentiable at $x = a$, then $x = a$ is an extremum when $f''(a) \neq 0$.

A. True

B. False

(Q7) This is called the second derivative test.

A critical point is not necessarily an extremum.

- The first derivative test fails:
  - $f(x) = x^3 \Rightarrow f'(x) = 3x^2$.
  - $f'(0) = 0$, but $f'(x) > 0$ for all other x!
- The second derivative test fails:
  - If $f(x) = x^4$, then $f''(0) = 0$ but $x = 0$ is a minimum.
  - If $g(x) = x^5$, then $g''(0) = 0$, but $x = 0$ is neither a maximum nor a minimum.

What about $f''(x)$?

Definition (Concavity)

- A function $f(x)$ is concave up on some interval if $f'(x)$ is increasing on that interval.
  - Q: Does it mean that $f''(x) > 0$?
  - When $f''(x)$ exists, this is the same as $f''(x) > 0$.
- A function $f(x)$ is concave down on some interval if $f'(x)$ is decreasing on that interval.
  - When $f''(x)$ exists, this is the same as $f''(x) < 0$.

Inflection points

Definition (Inflection point)

An inflection point of $f(x)$ is a point at which the concavity changes.

Q8. If $f''(a) = 0$, then $x = a$ is an inflection point of $f(x)$.

A. True

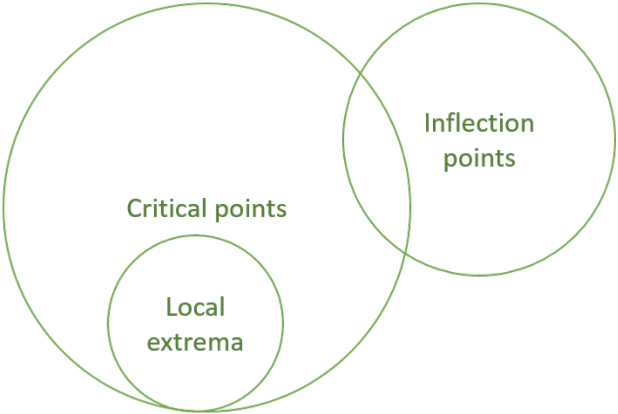
B. False

- If $f(x) = 2x$, then $f''(x) = 0$ for all x.

Example

Find the zeros, critical points, and inflection points of the function $g(x) = x^5 - x^3$.

Summary I



Note: here by local extrema we don't consider end points of intervals

Summary II

Fill in the blanks (if an answer exists)

	Concave up	Concave down
Increasing		
Decreasing		

Summary III

- Definition of monotonicity
  - If f' exists, it can be used to identify monotonicity.
- Definition of critical points and local extrema
  - If f' exists, FDT may identify local extrema, from the change of sign of f'. (sufficient but not necessary condition)
  - If f'' exists, SDT may identify local extrema, from the sign of f''. (sufficient but not necessary condition)
- Definition of concavity and inflection points

Answers

1. D
2. A
3. A B and C
4. B
5. A
6. A
7. A
8. B

Related exam problems

1. Find the zeros, critical points, and inflection points of the function
$$f(x) = -x^4 - 2x^3.$$
2. Find all minima, maxima, and infection points of $f(x) = x^4 - x^2$. Sketch the graph of $f(x)$.
3. Consider the function $f(x) = x^4 + ax^3 - x^2$. Find all values of a for which f has no inflection points, or show that no such values exist.