Root finding: Newton's method

Math 102 Section 102
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Last time

- Sketching anti-derivatives (only determined up to a vertical shift).
- Tangent line equations. Two key things:
  - Which point?
  - What slope? (derivative)
- Linear approximation using a tangent line.
  - Which is the "anchoring" point? Which is the point you want to approximate?

Today

Newton's method: find an approximation for the zero of a smooth function (called the root of the function).

- Very useful in science, engineering, finance etc.

You have all the knowledge needed to construct such a method. Let's work it out together!

- Geometric view (Doc cam picture will be uploaded.)

Idea behind Newton's method

- What's the value of x^* such that $f(x^*) = 0$? Start with (an arbitrary) initial guess x_0.
- Use the tangent line to replace the original function at x_0:

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

- Find the point x_1 where the tangent line crosses the x-axis:

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

- Solve for the approximate root x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Idea behind Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The approximate value (possibly) does not work that well. What should we do?

- Now we have an updated guess x_1, which is hopefully better than our initial guess x_0.
- Repeat this procedure multiple times! (called iterations)
- ~~And keep our fingers crossed~~.
- In many cases, $x_k \to x^*$ as $k \to \infty$.

Newton's method

Algorithm (Newton's method)

Given an approximation x_k for the root of the equation $f(x) = 0$, we can improve the accuracy of that approximation with another iteration using Newton's Method:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Newton's method

Example (Root finding with Newton's method)

Use three iterations of Newton's method to approximate a root of

$$f(x) = x^3 - 2x + 1$$

with $x_0 = 0$.

Newton's method

Geometric views:

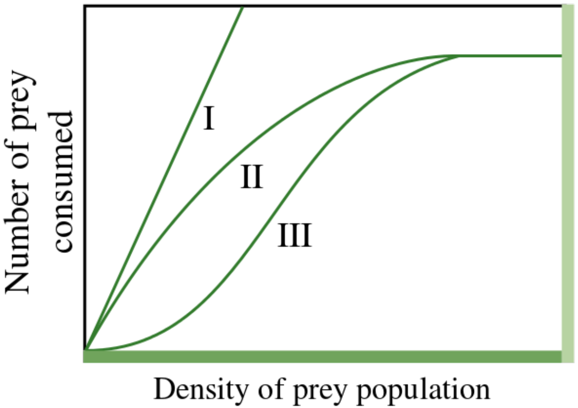
- Desmos demo:
  https://www.desmos.com/calculator/pshtlnokny

Observations:

- Iterative method: use successive approximations to approach the exact solution.
- Convergence is fast (if it converges).
- Local convergence only.

Example: predator-prey

A revisit to the predator-prey model: three types of predator response to prey population



Example: predator-prey

- $x =$ number of the prey
- rate of predation $P(x)$ by the predator is

$$P(x) = \frac{30x^3}{20^3 + x^3}$$

- The prey reproduce at the rate

$$G(x) = 0.5x$$

- Is there a population size x such that the predation rate exactly balances the reproduction rate?

Example: predator-prey

For which density x does $P(x) = G(x)$?

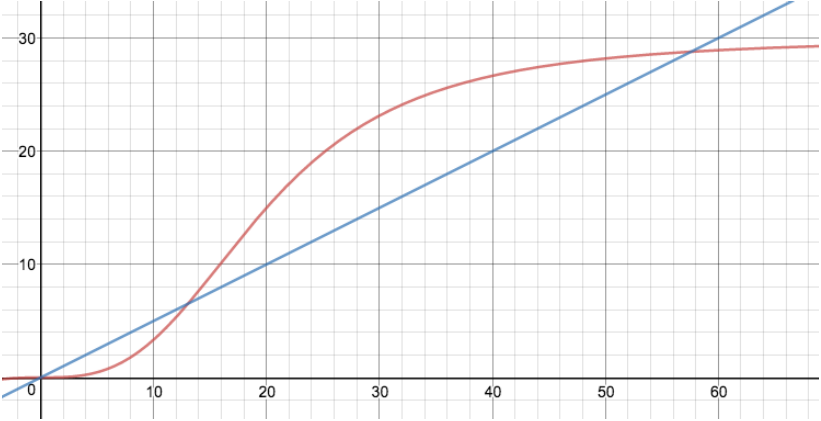
$$P(x) = \frac{30x^3}{20^3 + x^3}, \quad G(x) = 0.5x$$

Set $P(x) = G(x)$

$$\begin{aligned} \frac{30x^3}{20^3 + x^3} &= 0.5x \\ \frac{60x^2}{20^3 + x^3} &= 1 \\ 60x^2 &= 20^3 + x^3 \\ x^3 - 60x^2 + 20^3 &= 0 \end{aligned}$$

Hard to solve. Use Newton's method (spreadsheet).

Example: predator-prey



Predation balances with prey reproduction at intersection points $(P(x) = G(x))$

Example: approximate value of functions

Example

Use Newton's method to approximate $\sqrt{50}$.

Q1. Key idea: set

A. $f(x) = \sqrt{x} - \sqrt{50}$

B. $f(x) = \frac{x}{\sqrt{50}} - 1$

C. $f(x) = x^2 - 50$

D. $f(x) = x - \sqrt{50}$

Today

Newton's method: an approximation method to find zeros of functions

- Iterative method: use successive approximations to approach the exact solution.
- Convergence is fast (if it converges).
- Local convergence only.

Answers

1. C

Related exam problems

1. Find an approximation for $\sqrt{105}$ using Newton's Method. Find x_1 only.