Antiderivative sketching, tangent lines and linear approximation

Math 102 Section 102
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Sep. 26, 2018

Annoucements

- Midterm practice problems
- Midterm Q&A sessions
- Take your own responsbility!

Last time

- Cats (anti-derivatives)
- Sea otters and others (compound function)
- Chain rule

Today

- Sketching anti-derivatives
- Tangent lines
- Linear approximation

Anti-derivatives

Q1. If $f'(x) = 6x^2 + 4x - 1$, what is a possible $f(x)$?

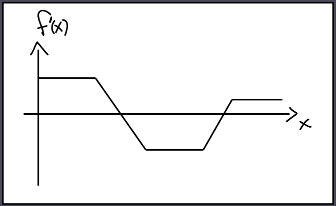
A. $f(x) = 12x + 4$

B. $f(x) = 2x^3 + 2x^2 - x$

C. $f(x) = 2x^3 + 2x^2 - x + 2$

D. $f(x) = 2x^3 + 2x^2 - x + C$

Slopes at each x don't change with a vertical shift!



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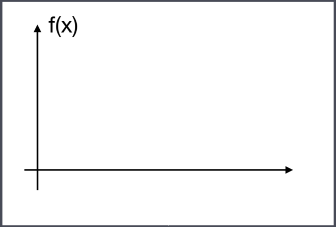

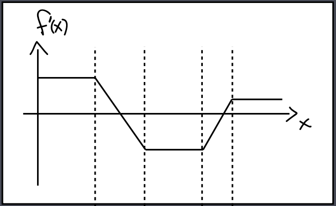

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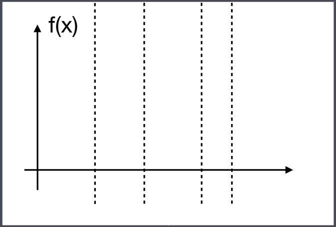

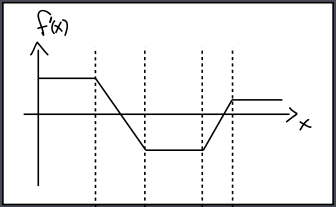

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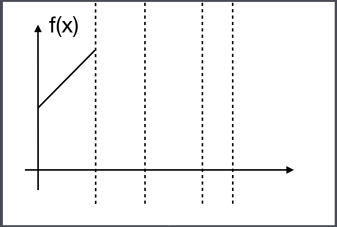

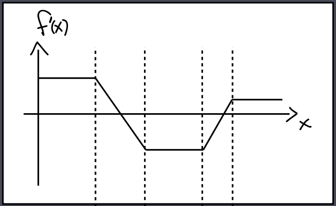

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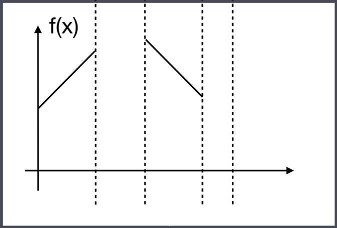

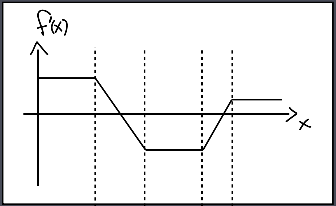

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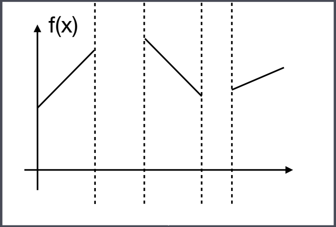

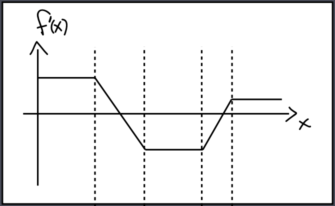

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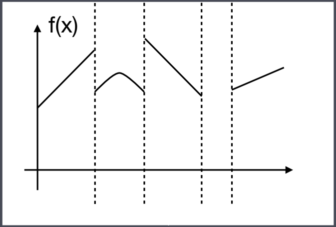

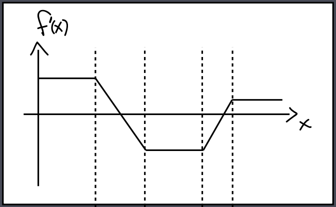

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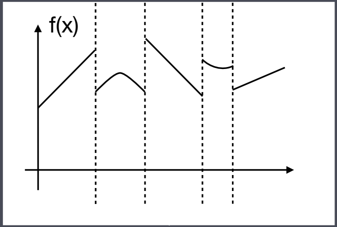

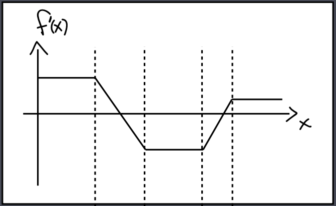

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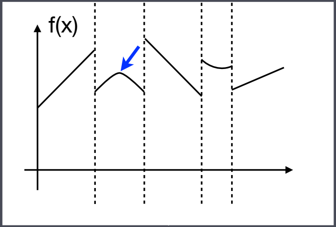

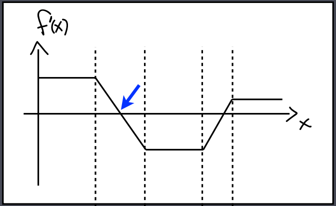

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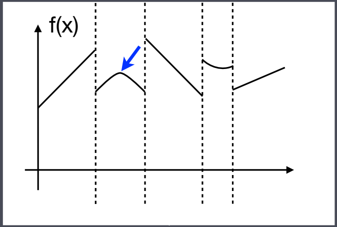

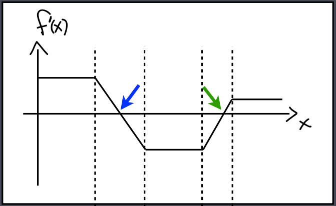

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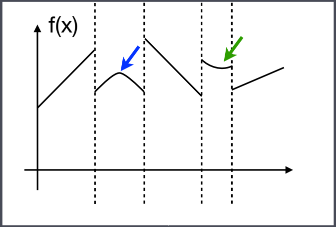

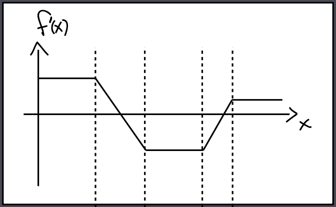

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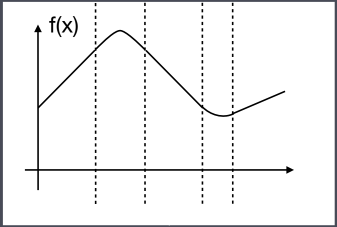

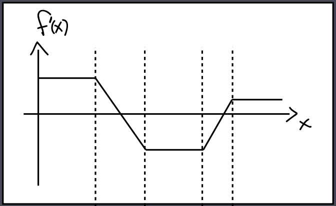

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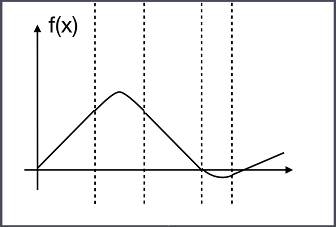


Only determined up to a vertical shift.

Back to tangent lines

- Our discussion of antiderivatives was a gentle introduction to a later topic (differential equations).
- What else can we do with tangent lines?
  - Approximate a smooth function using a straight line (today)
  - Approximate the zeros of a smooth function — Newton's method (Friday)

Tangent lines

- The tangent line at a point is the unique line that looks like the function when you zoom in on that point.
- Slope of tangent line $= f'(a)$ at point $x = a$
- Equation of tangent line at $(a, f(a))$:

$$y - f(a) = f'(a)(x - a).$$

Typical tangent line problem

Example

Find a tangent line to the function $f(x) = x^3 + 2x^2 - x + 2$ that is parallel to the line $y = -x + 3$.

- We need to find a such that $f'(a) = -1$.
- $f'(a) = 3a^2 + 4a - 1 = -1$
- $3a^2 + 4a = 0$
- $a = 0, \frac{-4}{3}$
- Tangent line 1: $y = -x + 2$
- Tangent line 2: ?

Exercise

Find equations of all tangent lines to $f(x) = x^2$ that go through the point $(-1, -1)$.

Solution to exercise

Derivative:

$$f'(x) = 2x.$$

Let (a, a^2) be a point on the graph of the function. The equation for the tangent line (point-slope form) is:

$$y - a^2 = 2a(x - a).$$

We know that the line goes through $(-1, -1)$. Plug the point in to figure out a:

$$-1 - a^2 = 2a(-1 - a).$$

After some manipulation:

$$a^2 + 2a - 1 = 0.$$

Recall the formula to solve a quadratic equation. We get

$$a = -1 \pm \sqrt{2}.$$

So there are two tangent lines of the original function passing through the point $(-1, -1)$. Now complete the equations of them...

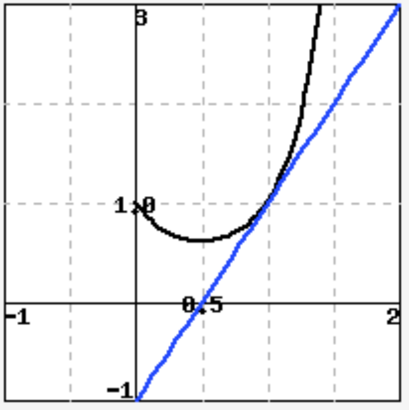
Linear approximation

Example (PL 4.2)

Find an approximation for $f(1.5)$

Q2. What is the point of tangency?

A. (0,-1)
B. (2,1)
C. (1,1)
D. (1,2)



Linear approximation (local approximation using tangent line)

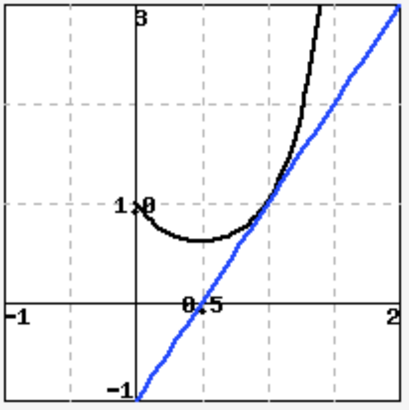
Q3. What is the slope of the tangent line?

A. 2

B. 3

C. 3/2

D. -3



Linear approximation

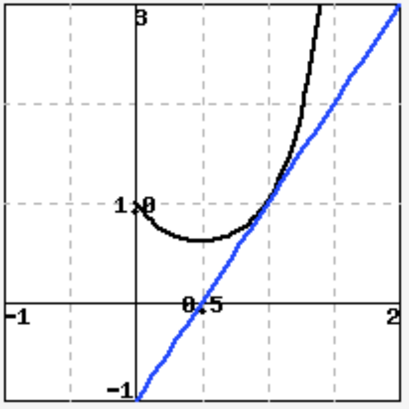
Q4. What is the equation of the tangent line?

A. $y = 2x - 1$

B. $y = 2x + 1$

C. $y = 2x - 0.5$

D. $y = x - 1$



Linear approximation

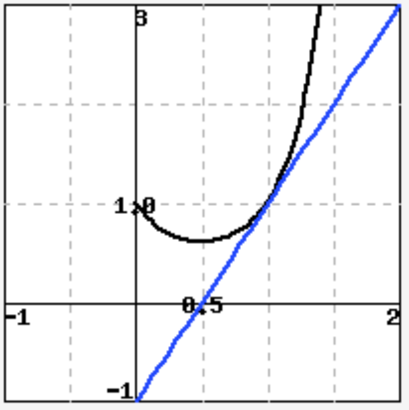
Q5. The linear approximation to $f(1.5)$ is

A. $f(1.5) \approx 1$

B. $f(1.5) \approx 1.5$

C. $f(1.5) \approx 2$

D. $f(1.5) \approx 2.5$

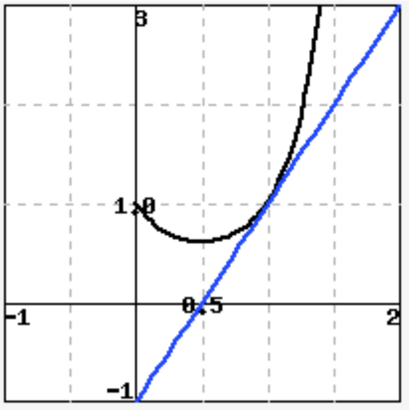


Linear approximation

Q6. The linear approximation to $f(1.5)$ is

A. an overestimate

B. an underestimate



Note: this would be more clear later in the term, when we talk about concavity of functions.

Linear approximation (general)

- Want $f(b)$, but it's too hard to calculate
- Know $f(a)$ and $f'(a)$, and these are easy to calculate
- Use the tangent line (linear approximation) to approximate $f(b)$

Fact (Local approximation using the tangent line)

Given a smooth function $f(x)$ and a point $(a, f(a))$ on its graph, one can approximate the value of the function close to the point $x = a$ using its tangent line at $x = a$:

$$L(x) = f(a) + f'(a)(x - a).$$

Note: this is also the important foundation in our next lecture for Newton's method of finding zeros of a function.

Estimate $\sqrt{99}$

Q7. Find the tangent line to $f(x) = \sqrt{x}$ at

A. $x = 1$

B. $x = 10$

C. $x = 99$

D. $x = 100$

Estimate $\sqrt{99}$

Q8. Plug ? into the linear approximation

$$L(x) = f'(a)(x - a) + f(a)$$

A. $x = 1$
B. $x = 10$
C. $x = 99$
D. $x = 100$

Estimate $\sqrt{99}$

Q9. The estimate will be an

A. overestimate

B. underestimate

Estimate $\sqrt{99}$

Q10. No calculators allowed. $\sqrt{99} \approx$

A. 9.94

B. 9.95

C. 9.96

D. 9.97

E. Help!

Estimate $\sqrt{99}$

- $f(x) = \sqrt{x} = x^{\frac{1}{2}}$.
- $x = 99$
- $a = 100$

$$
\begin{aligned}
L(x) &= f(a) + f'(a)(x-a) \\
L(99) &= f(100) + f'(100)(99-100) \\
&= 10 + \frac{1}{20}(-1) \\
&= 9.95
\end{aligned}
$$

Today

- Sketching anti-derivatives (only determined up to a vertical shift).
- Tangent line equations. Two key things:
  - Which point?
  - What slope? (derivative)
- Linear approximation using a tangent line.
  - Which is the "anchoring" point? Which is the point you want to approximate?

Answers

1. B, C, or D
2. C
3. A
4. A
5. C
6. B
7. D
8. C
9. A
10. B

Related exam problems

1. Consider the function $f(x) = \frac{3}{x-2}$. At which points $(a, f(a))$ does the graph of this function have tangent lines parallel to the line $y = -x$. What is the equation of the tangent lines at each of these points?
2. Find the point or points on the graph of f whose tangent line to f goes through the point $(0, 1/2)$ where $f(x) = \frac{1}{1+x^2}$.

Related exam problems

3. Find a linear approximation that for $(1.1)^8$. Explain why the approximation is an over or under estimate.