Anti-derivatives and chain rule

Math 102 Section 102
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Due due due due due...

- Sep 24 (Today): WeBWork Pre-lecture 4.1
- Sep 26 (Wed): WeBWork Pre-lecture 4.2
- Sep 27 (Thu): WeBWork Assignment 3
- Sep 28 (Fri): OSH 2 (start early!)

Last time

- Power rule, product rule, quotient rule

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}$$

- Linear operators
- Anti-derivatives

Today

- Anti-derivative and cats
- Compound functions
  - Sea otters, sea urchins, kelp forests
- Chain rule
- Practice

Antiderivatives & cats

Cats tend to land on their feet when falling (0.3 seconds to flip over).

Question

What is the minimum height from which a cat can fall to ensure it lands on its feet?

Annals of Improbable Research
Diamond, J. (1988) *Why cats have nine lives*, Nature 332.
Thanks to Joseph M. Mahaffy for the falling cat idea.

Antiderivatives & cats

Idea: let $h(t)$ be the height of the cat above ground at time t, and suppose $h(0) = h_0$ (pronounced as "h-naught"). If $h(0.3) \geq 0$, then the cat is safe.
Newton's second law of motion provides:

$$ma = -mg$$

- m is the mass of the cat
- a is the acceleration of cat
- $-mg$ is the force of gravity ($g = 9.81$ m/s^2)

Hence our model

$$h''(t) = a(t) = -g.$$

Initially

- $h(0) = h_0$
- $v(0) = h'(0) = 0$ (initial velocity is zero)

Antiderivatives & cats

Use antiderivatives:

- Acceleration: $h''(t) = -g$
- Velocity: $h'(t) = -gt + B$
- Height: $h(t) = -g\left(\frac{t^2}{2}\right) + Bt + A$
- Initial height: $h(0) = h_0 \Rightarrow A = h_0$
- Initial velocity: $h'(0) = v_0 \Rightarrow B = v_0 = 0$

Height of cat above ground:

$$h(t) = -g\left(\frac{t^2}{2}\right) + h_0$$

Antiderivatives & cats

The height of the cat at time 0.3 seconds is $h(0.3)$:

$$h(0.3) = \frac{-g(0.3)^2}{2} + h_0 = -0.441 + h_0,$$

so the cat must fall from a height of at least 44.1 cm to have sufficient time to flip over before hitting the ground.

Note: This is consistent with the Annals of Improbable Research data, which suggests that a cat dropped upside down from a height of 2 to 6 feet (≈60 cm to ≈ 180 cm) will always land on its feet.

Sea otters, urchins, and kelp

Sea otter is a so-called "keystone species" whose population significantly affects a lot of others. Watch the video clip and note the relationship between sea otters and two other species. Video: 2:27-3:50

If you have not seen sea otters, go to Vancouver Aquarium! You don't get to see them everywhere!

Composition of functions

- t: time
- o: otter population
- u: urchin population
- k: kelp population

$$k = k(u) = k(u(o)) = k(u(o(t))).$$

Q1. $k'(u)$ is

A. positive

B. negative

C. not sure

Q2. $k'(o)$ is

A. positive

B. negative

C. not sure

Composition of funcitons

Q3. If $f(x) = 2x + 3$ and $g(x) = -4x + 2$,

A. $h(x) = f(g(x)) = -8x + 7$

B. $h(x) = f(g(x)) = -8x - 10$

C. $h(x) = f(g(x)) = -2x^2 - 8x + 6$

D. $h(x) = f(g(x)) = -8x + 5$

Solution:

$$h(x) = f(g(x)) = 2(-4x + 2) + 3 = -8x + 7$$

Composition of functions:

$$f \circ g(x) = f(g(x))$$

Chain rule

Fact (Chain rule)

Suppose f and g are differentiable functions, then

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = \frac{df}{dg}\frac{dg}{dx}.$$

Chain rule

Example

Differentiate $h(x) = (x^2 + x^3)^4$

Let $f(x) = x^4$ and $g(x) = x^2 + x^3$

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 4(x^2 + x^3)^3 \cdot (2x + 3x^2) \\ &= 4(2x + 3x^2)(x^2 + x^3)^3 \end{aligned}$$

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Answers

1. B
2. A
3. A

Related exam problems

1. Suppose that $f(x) = (3 + x^2)g(x)$ and that $g(2) = 2$, $g'(2) = -1$. Compute $f'(2)$.
2. Find the equation of the tangent line to the graph of $f(x) = x^3 - 3x + 1$ at $x = 0$. Where does the tangent line intersect the x-axis?