Derivatives and anti-derivatives

Math 102 Section 102
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Spreadsheet example

- Why? Only need $f(x)$ values to construct an approximation.
- How? Use secant lines to approximate tangent lines (finite difference approximation).
- The accuracy depends on Δx.
- By programming features, the spreadsheet can calculate the derivative at most of the points automatically.

Make a copy of my sheet or make up your own, and play around:
https://docs.google.com/spreadsheets/d/11l0hj_k5IBrj03ayFXtN4Q4khKs0l1996tR5L8v0h6g/edit?usp=sharing

Last time

- Algebraic skills in computing limits and derivatives
- More sketching examples. Key ideas:
  - Maxima/minima correspond to zeros in derivative
  - Signs of slope
- Going backwards: given $f'(x)$, sketch $f(x)$
- Position, velocity and acceleration and the second derivative
- Approximating the derivative computationally (an introduction to scientific computing)

Today

- Properties of derivatives
  - Power rule
  - Linear operations
  - Antiderivatives
  - Product rule
  - Quotient rule

Power rule

Derivatives easily computed using the definition:

$f(x)$	$f'(x)$
1	0
x	1
x^2	$2x$
x^3	$3x^2$
$\vdots$	$\vdots$
x^n	nx^{n-1}

Power rule

Fact (Power rule)

For any integer n,

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

- To show this from definition, you need the fancy algebra of binomial expansion. If you happen to know this, try it. (not required for exam.)
- This also holds for negative and fractional powers!

Linear operator

Definition (Linear operator)

An operator is a transformation of a function into another. An operator L is said to be linear if for any function f, g and any constant c,

$$L\left(f(x)+g(x)\right)=L\left(f(x)\right)+L\left(g(x)\right),$$

and

$$L\left(cf(x)\right)=cL\left(f(x)\right).$$

Linearity of the derivative

Fact (Linearity of the derivative)

The derivative is a linear operation:

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{df}{dx} + \frac{dg}{dx}$$

and

$$\frac{d}{dx}(cf(x)) = c\frac{df}{dx}$$

for some constant c and differentiable functions f and g.

Can be shown easily by definition. Try it.

Derivatives of polynomials

Q1. The derivative of $f(t) = At^2 + Bt + C$ is

A. $f'(t) = 2At + B$

B. $f'(t) = At + B$

C. $f'(t) = 2t + B + C$

D. $f'(t) = C$

(thanks to the power rule and the linearity of the derivative)

Higher derivatives

- $f(t) = At^2 + Bt + C$
- $f'(t) = 2At + B$
- $f''(t) = 2A$
- $f'''(t) = f^{(3)}(t) = 0$

(notice the notation for nth-derivative: $f^{(n)}(t)$)

Higher derivatives

Q2. Which of the following is always true for a cubic polynomial function?

A. $f(x) = 0$

B. $f'(x) = 0$

C. $f''(x) = 0$

D. $f'''(x) = 0$

E. $f^{(4)}(x) = 0$

Antiderivatives of polynomials

Q3. The anti-derivative of $f'(t) = Ct + B$ is

A. $f(t) = C$

B. $f(t) = Ct^2 + Bt + A$

C. $f(t) = C\left(\frac{t^2}{2}\right) + Bt$

D. $f(t) = C\left(\frac{t^2}{2}\right) + Bt + A$

(which $f(t)$ has $f'(t) = Ct + B$? Work backwards!)

The anti-derivative of a function is only defined up to an additive constant. (moving the curve up and down, but with the same shape)

Product rule

Fact (Product rule)

If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)g(x)$ is also differentiable and

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

Fact (Quotient rule)

If $f(x)$ and $g(x)$ are differentiable functions and $g(x) \neq 0$, then $f(x)g(x)$ is also differentiable and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Quotient rule

Example (Quotient rule)

If $f(x) = \frac{x^3}{1+x^3}$, then

$$f'(x) = \frac{(1+x^3)3x^2 - x^3(3x^2)}{(1+x^3)^2}$$
$$= \frac{3x^2}{(1+x^3)^2}.$$

Quick check

True or false?

1. $$\frac{d}{dx}(7x^3) = \frac{d}{dx}(7) \cdot \frac{d}{dx}(x^3) = 0 \cdot 3x^2 = 0$$

2. $$\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = x \cdot \frac{d}{dx}(x) = x \cdot 1 = x$$

3. $$\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x)\frac{d}{dx}(x) = 1 \cdot 1 = 1$$

Thinking mathematically

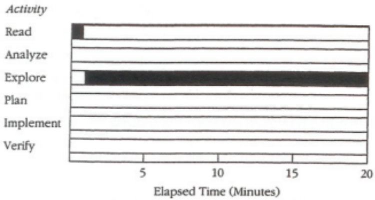

FIGURE 15–3. Time-line graph of a typical student attempting to solve a non-standard problem.

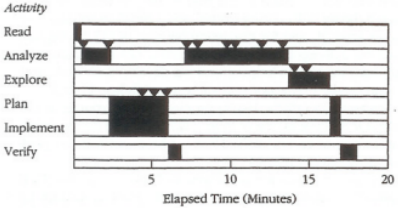

FIGURE 15–4. Time-line graph of a mathematician working a difficult problem.

Facing long problems

Keep in mind[1]:

- **What (exactly) are you doing?** (Can you describe it precisely?)
- **Why are you doing it?** (How does it fit into the grand scheme of solving your problem?)
- **How does it help you?** (What will you do with the outcome when you obtain it?)

[1] Alan H. Schoenfeld, Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics, 1992

Today

- Linear operators
- Power rule, product rule, quotient rule
- Antiderivatives
- Mathematical thinking

Related exam problems

1. (Quotient Rule) At an all-you-can-eat buffet, the total calories you gain can be represented by the function

$$E(t) = \frac{At}{b+t},$$

where $t \geq 0$ is the time in minutes you spend at the restaurant and A and b are positive constants.

- If you stayed for a long time, what asymptote would your total caloric gain approach?
- After how much time do you gain exactly half of that asymptotic caloric amount?
- At time t, what is the instantaneous rate at which your caloric gain changes?

Answers

1. A
2. E
3. D

[True or False?] All false