The derivative:
a geometric view

Math 102 Section 102
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Announcement

- Computers welcome on Wednesday: computations using spreadsheets

Last time

- Continuous functions and types of discontinuity
- Which of the following describes you?
  A. I was working too hard on the practice problems and did not notice any joke.
  B. I read the joke and worked on the practice problems.
  C. I read the joke but did not do the practice problems.
  D. What the hell is going on? Am I in the right classroom?

Exercise

Q5 (last time). What value of a makes the following function continuous?

$$f(x) = \begin{cases} ax^2 + 1, & x < 1 \\ -x^3 + x, & x \geq 1. \end{cases}$$

A. $a = 1$

B. $a = 0$

C. $a = -1$

D. no value of a works

Exercise

Q5 (last time). What value of a makes the following function continuous?

$$f(x) = \begin{cases} ax^2 + 1, & x < 1 \\ -x^3 + x, & x \geq 1. \end{cases}$$

Tip: set

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(a).$$

https://www.desmos.com/calculator/v8hfkabrry

Today

- Derivative, cont'd
    - Geomeric perspective
    - Computational perspective (next time)

Derivative

Definition (Derivative)

The derivative of a function $y = f(x)$ at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We can also write $\left.\frac{df}{dx}\right|_{x_0}$ to denote $f'(x_0)$

- Tip: definition is important in mathematics.

Example

Derivative of $f(x) = x^3$. (left as exercise)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2
\end{aligned}$$

Derivative: geometric view

Fact (Local approximation to a smooth function using a line)

Any smooth function looks like a (unique) straight line when zooming in. This line is the tangent line of the function. (This can also be taken as the definition for the tangent line.)
In other words, locally one can use the tangent line to approximately represent a smooth function.

- https://www.desmos.com/calculator/mekfho0w38
- Experiment with the slider.
- Then experiment with zoom!

Tangent lines

For smooth functions:

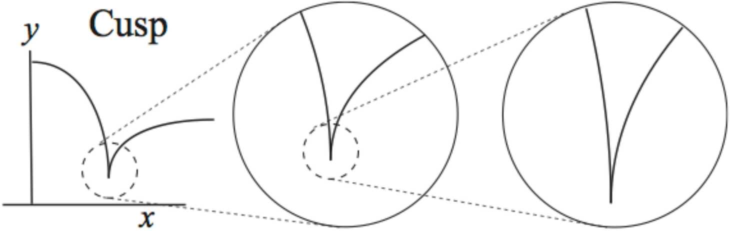
secant line		tangent line
slope $= \frac{f(a+h)-f(a)}{h}$	$\xrightarrow{h\to 0}$	slope $= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$
average rate of change		instantaneous rate of change

The derivative tells us how fast the function is changing.
Equation of tangent line at $(a, f(a))$:

$$y - f(a) = f'(a)(x - a).$$

Tangent lines

- Tangent lines do not always exist, for example, functions with cusps:



Differentiability and continuity

- Continuity = is continuous
- Differentiability = has a derivative

Q1. A function that is continuous at a point must also be differentiable at that point?

A. True

B. False (For example, $f(x) = |x|$)

Q2. A function that is differentiable at a point must also be continuous at that point?

A. True (Differentiable $\Rightarrow$ looks like tangent line)

B. False

Analytic argument:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

must exist.

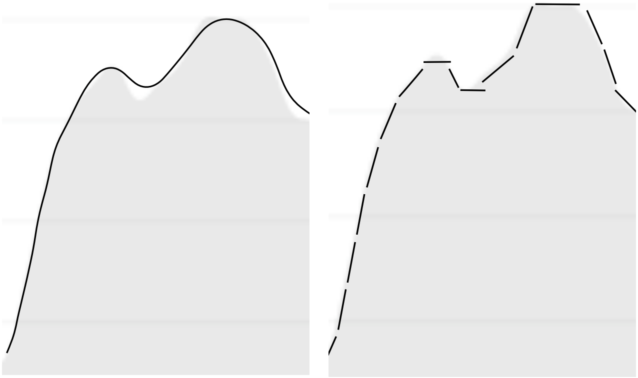
Sketching derivatives

- Elevation data:

Sketching derivatives

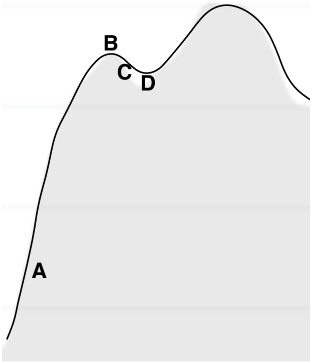
Zoom in on some data:

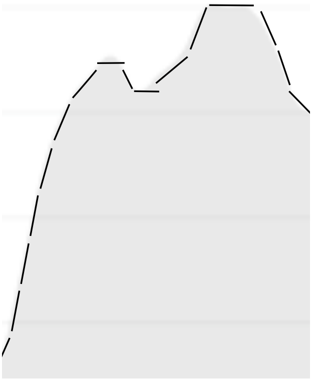
Sketching derivatives



Sketching derivatives

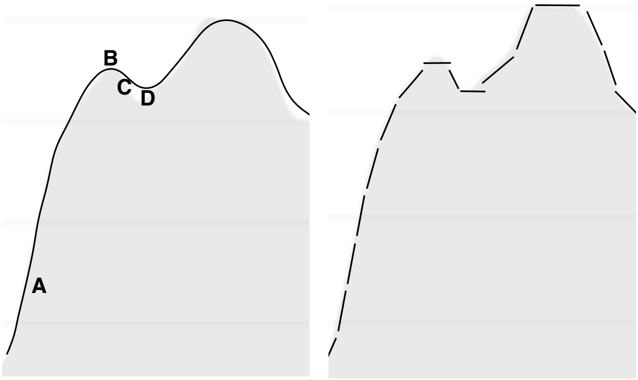
Q3. Where is the slope greatest?





Sketching derivatives

Q4. Where is the slope zero?

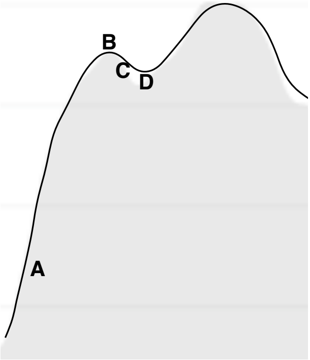


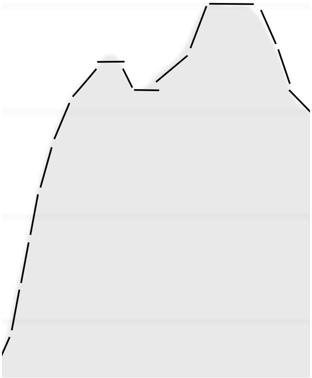
Sketching derivatives

Q5. Is that the only place where the slope is zero?

A. No

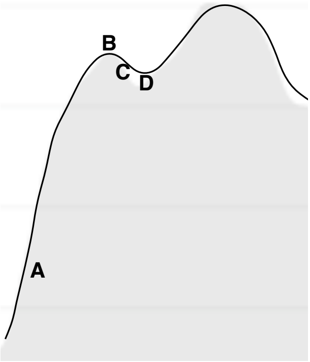
B. Yes

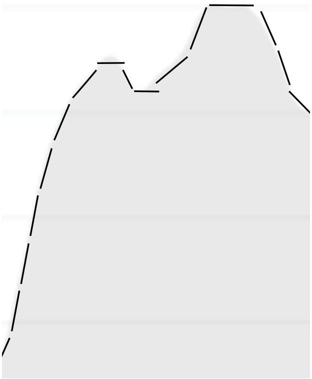




Sketching derivatives

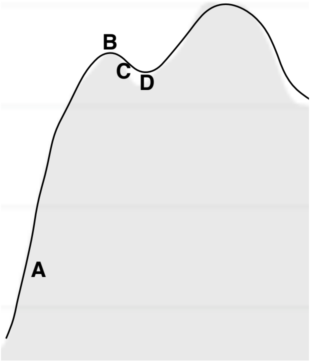
Q6. Where is the slope negative?





Sketching derivatives

Sketch the derivative of the function below on the board or document camera. (Graph scanned and uploaded separately.)



For you to think about: why is a local maximum/minimum always at $f'(x) = 0$? Is this true the other way around?

Sketching derivatives

Q7. Identify the function shown in the graph on the right.

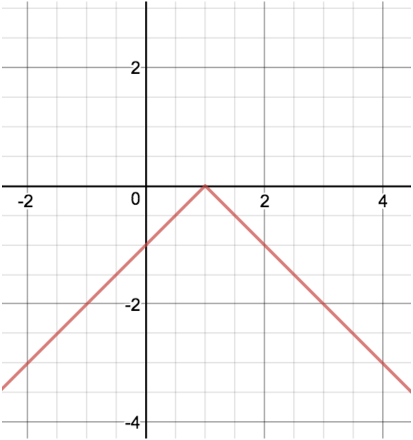
A. $|x+1|$

B. $|x-1|$

C. $-|x+1|$

D. $-|x-1|$

E. $-(x+1)$



Sketching derivatives

Q8. This function is

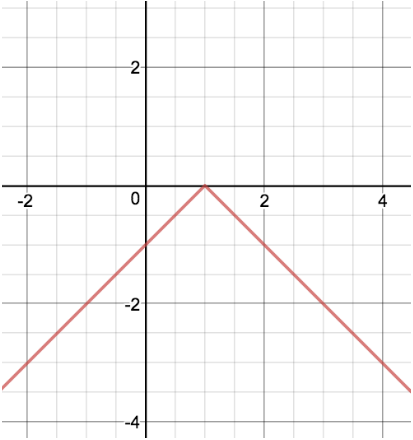
A. discontinuous

B. not differentiable anywhere

C. has no limit as $x \to 0$

D. differentiable except at $x = 1$

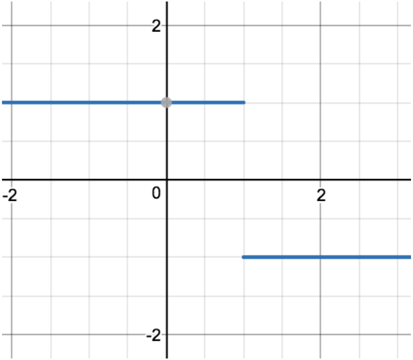
E. I am confused



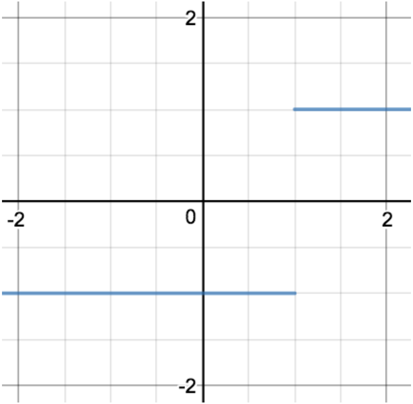
Sketching derivatives

Q9. The derivative of this function is

A.



B.



(with open circles on all lines at $x = 1$.)

Today...

- Derivative: how fast a function is changing
- Tangent lines: zoom into a smooth graph
- Differentiability implies continuity, but continuity does not imply differentiability
- Sketching derivatives
  - Think about slopes
  - Maxima/minima of function correspond to zeros of derivative
- Do the following graphing activity in a small group of 2-3 people...

Graphing activity

To do:

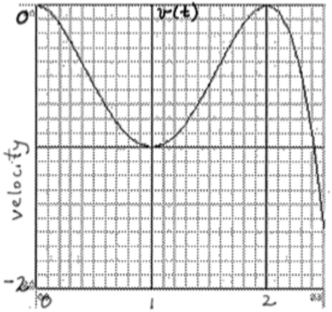
- Graph $f(x) = |\sin(x)|$.
- Graph the derivative $g(x) = f'(x)$.
- Find the equation of the tangent line at $x = a$.
- Draw the tangent line on your graph.

Answers

1. B
2. A
3. A
4. B or D
5. A
6. C
7. D
8. D
9. A

Related exam problems

1. Use the definition of the derivative to compute the derivative of $f(x) = \frac{1}{x+1}$.
2. Shown in the graph is the velocity of a particle. Use this to sketch the acceleration of the particle.



The joke page

OK I guess most of you now expect a joke at the end. You are not allowed to read this until you finish the exercises in the above slides.

Why did the two 4s skip lunch? They already 8!