Limits, continuity, derivative and examples
Math 102 Section 102

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Annoucements

- WebWork attempts
- Group work on OSH (size ≤ 4)
- MLC and OSH
- Add/drop class: Tue, Sep 18; with a "W" afterwards
- Regrading homework
- Pace: fast or slow?

Last time

- Average rate of change
- Limits

Definition (Limit of a function)

Let $f(x)$ be a function, and a is a constant. When x approaches a, if $f(x)$ becomes closer and closer to a number, represented by c, then we say the limit of $f(x)$ at $x = a$ is c, denoted by

$$\lim_{x \to a} f(x) = c.$$

  - Some limits DNE
  - plug-in for continuous functions
  - factoring
- Finish discussions on instantaneous rate of change...

Today

- Continuous functions
- Types of discontinuities
- Examples: practice!

Continuous function

Definition (Continuous function)

A function $f(x)$ is continuous at a point $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

- Continuity is a local property. "$x \to a$"
- For $f(x)$ to be continuous at $x = a$, three things need to be true:
  1. $f(x)$ needs to be defined at $x = a$.
  2. The limit of the function exists as x approaches a.
  3. The limit value is the same as the value of the function at a.

Types of discontinuities

Q1. The function

$$f(x) = \frac{x^3 - ax^2}{x - a}$$

is

A. Continuous at $x = a$

B. Discontinuous at $x = a$

C. Continuity at $x = a$ depends on the value of a

D. None of the above

Types of discontinuity

$$f(x) = \frac{x^3 - ax^2}{x - a} = \begin{cases} \frac{x^2(x-a)}{x-a} = x^2, & \text{if } x \neq a; \\ \text{undefined}, & \text{if } x = a. \end{cases}$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2.$$

reminder: when talking about limits, we consider x values *close* to a, but not equal to a.

- Hole-in-graph discontinuity

Types of discontinuities

Q1.1. The function

$$f(x) = \frac{x^3 - ax^2}{x - a}$$

is undefined at at $x = a$. What value should be assigned to $f(a)$ such that $f(x)$ is continuous at $x = a$?

A. 0

B. a

C. a^2

D. a^3

Types of discontinuities

Q1.1. The function

$$f(x) = \frac{x^3 - ax^2}{x - a}$$

is undefined at at $x = a$. What value should be assigned to $f(a)$ such that $f(x)$ is continuous at $x = a$?
Define

$$f(a) = \lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^3 - ax^2}{x - a}$$
$$= \lim_{x \to a} x^2 \frac{x - a}{x - a} = \lim_{x \to a} x^2 = a^2$$

- It's a removable discontinuity.

Types of discontinuities

Q2. Consider the function

$$g(x) = \begin{cases} -1, & x < b, \\ 1, & x \geq b. \end{cases}$$

What is $\lim_{x \to b} f(x)$?

A. 1

B. -1

C. 0

D. The limit does not exist

Types of discontinuities

Q2. Consider the function

$$f(x) = \begin{cases} -1, & x < b, \\ 1, & x \geq b. \end{cases}$$

$\lim_{x \to b} f(x)$ DNE since the right-side and left-side limits are not equal:

- Approach $x = a$ from above (right limit):

$$\lim_{x \to a^+} f(x) = 1$$

- Approach $x = a$ from below (left limit):

$$\lim_{x \to a^-} f(x) = -1$$

- Jump discontinuity

Types of discontinuities

Q3. What is

$$\lim_{x \to a} \frac{1}{x-a}?$$

A. ∞

B. $-\infty$

C. DNE

D. More than one answer is correct

Types of discontinuities

Q3. What is

$$\lim_{x\to a} \frac{1}{x-a}?$$

- The function $f(x) = \frac{1}{x-a}$ is not defined at $x = a$.
- $\lim_{x\to a} \frac{1}{x-a}$ DNE:
  - Right-side limit:

$$\lim_{x\to a^+} \frac{1}{x-a} = +\infty \quad (x > a \text{ means } x - a > 0).$$

  - Left-side limit:

$$\lim_{x\to a^-} \frac{1}{x-a} = -\infty \quad (x < a \text{ means } x - a < 0)$$

- Blow-up discontinuity

Types of discontinuities

- Hole in graph (removable)
  - Left and right limits exist and are equal, but the function lacks definition on the spot.
- Jump
  - Left and right limits exist, but are not equal.
- Blow up
  - At least one of left or right limits DNE.
- Other types
  - Left or/and right limits DNE, not necessarily due to blow-up.

https://www.desmos.com/calculator/viztz0quu8

Questions?

If not, I have many more questions.

Q4. Which of the following limits DNE?

A.
$$\lim_{x\to 0} \frac{1}{x+1}$$

B.
$$\lim_{x\to -1} \frac{1}{x+1}$$

C.
$$\lim_{x\to -1} \frac{x+1}{x+1}$$

D. more than one of the above limits DNE

Q5. What value of a makes the following function continuous?

$$f(x) = \begin{cases} ax^2 + 1, & x < 1 \\ -x^3 + x, & x \geq 1. \end{cases}$$

A. $a = 1$

B. $a = 0$

C. $a = -1$

D. no value of a works

Q5. What value of a makes the following function continuous?

$$f(x) = \begin{cases} ax^2 + 1, & x < 1 \\ -x^3 + x, & x \geq 1. \end{cases}$$

Tip: set

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(a).$$

https://www.desmos.com/calculator/v8hfkabrry

Limits at infinity

Q6. Which of the following limits exist?

A.
$$\lim_{x\to\infty} x^3$$

B.
$$\lim_{x\to\infty} x^{\frac{1}{3}}$$

C.
$$\lim_{x\to\infty} 3^x$$

D.
$$\lim_{x\to\infty} x^{-3}$$

https://www.desmos.com/calculator/ihvvwf4zwh

Limits at infinity

Q7. What is the following limit?

$$\lim_{x\to\infty} \frac{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}{6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1}$$

A. 0

B. 6

C. $\frac{1}{6}$

D. DNE

Two methods: asymptotic approximation or factoring

Derivative

- The derivative of a function $y = f(x)$ at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- You can also write $\left.\frac{dy}{dx}\right|_{x_0}$ to denote $f'(x_0)$.

Derivative

Q8. To compute the derivative of the function $f(x) = (x+3)^2$, we need to compute:

A.

$$\lim_{h \to 0} \frac{(x+3+h)^2 - (x+3)^2}{h}$$

B.

$$\lim_{x \to 1} \frac{(x+3)^2 - x^2}{x}$$

C.

$$\lim_{h \to 1} \frac{(x+3+h)^2 - (x+3)^2}{h}$$

D.

$$\lim_{h \to 0} \frac{(x+3+h)^2 - (x+3)^2}{x}$$

Derivative

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h+3)^2 - (x+3)^2}{h} \\
&= \lim_{h\to 0} \frac{((x+3)+h)^2 - (x+3)^2}{h} \\
&= \lim_{h\to 0} \frac{(x+3)^2 + 2h(x+3) + h^2 - (x+3)^2}{h} \\
&= \lim_{h\to 0} \frac{2h(x+3) + h^2}{h} \\
&= \lim_{h\to 0} 2(x+3) + h \\
&= 2(x+3)
\end{aligned}
$$

Today & Challenge

- Continuous functions, limits, and the derivative.
- Practice! Practice! Practice!

Challenge Write the following limit as the derivative of a function:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

p.s.

- Definition is important in mathematics.
- How local is “local”? Can I zoom in infinitely? Short answer: yes.

Answers

1. B

1.1 C

2. D

3. C

4. B

5. C

6. D

7. C

8. A

Related exam problem

1. Use the definition of the derivative and the following hint to calculate the derivative of the function $f(x) = \sqrt{x}$. Hint:

$$\frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{(\sqrt{a} + \sqrt{b})} = \frac{a - b}{(\sqrt{a} + \sqrt{b})}$$

A bonus question

What happens if you put root beer in a square glass?
It just becomes beer.