Rate of change and derivatives
Math 102 Section 102

Mingfeng Qiu

Sep. 12, 2018

Annoucements

- Diagnostic test: if below 50%, talk to a faculty advisor
  - MATH 180, 184
  - MATH 110
- How do I find slides, office hours etc. ?
- Class reps, please come to see me.

Last time

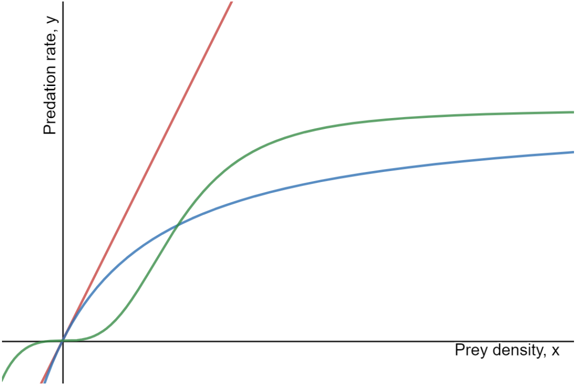
Hill functions

$$y = \frac{Ax^n}{a^n + x^n},\ x \geq 0,\ A, a > 0,\ n \geq 1$$

Last time

Example

The predation rate $P(x)$ of predators depends on prey density x.
There are three types of predators:

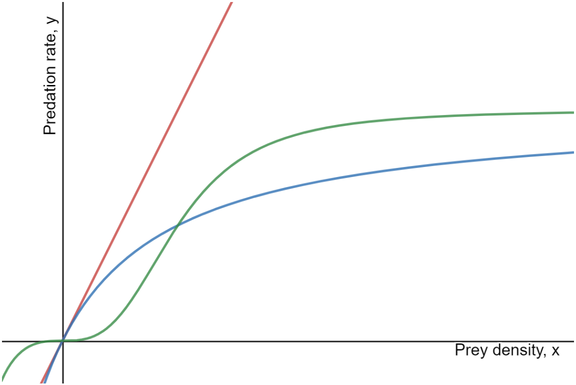


$$P(x) = Kx$$

$$P(x) = \frac{Kx}{a+x}$$

$$P(x) = \frac{Kx^n}{a^n + x^n}, \; n > 1$$

Predator response



$$P(x) = Kx$$

$$P(x) = \frac{Kx}{a + x}$$

$$P(x) = \frac{Kx^n}{a^n + x^n}, n > 1$$

I The more prey there is, the more I can eat.

II I get satiated and cannot keep eating more and more prey.

III I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

Learning goals

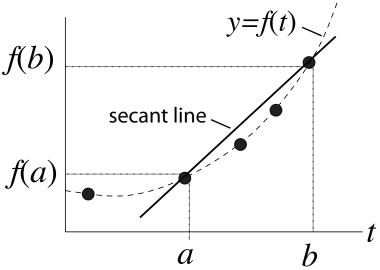
- Average rate of change and secant lines
- Instantaneous rate of change and tangent lines
- Limits
- Derivative

Average rate of change

Definition (Average rate of change)

Suppose that $y = f(t)$ is a function. The average rate of change of f over the interval $a \le t \le b$ is

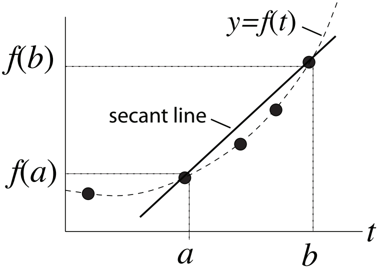
$$\frac{\text{Change in } f}{\text{Change in } t} = \frac{\Delta f}{\Delta t} = \frac{f(b) - f(a)}{b - a}$$



Secant line

Definition (Secant line)

The secant line to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q. The slope of the secant line gives the average rate of change of $f(x)$ from R to Q.



Secant line

Q1. A secant line is

A. A line whose slope is instantaneous velocity

B. A line connecting two points on a graph

C. The same as average velocity

D. The same all along the curve

E. Not sure

Average rate of change

Q2. Let $y = f(x)$ be a function. The average rate of change of f over the interval $x_0 \le x \le x_0 + h$ is

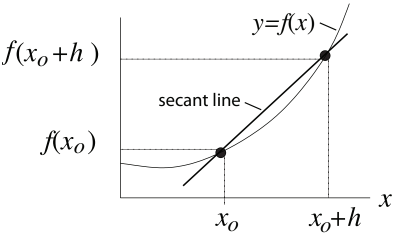
A. $\frac{f(x_0)-f(h)}{x}$

B. $\frac{f(x_0+h)-f(h)}{h}$

C. $\frac{f(x_0+h)-f(x_0)}{x_0}$

D. $\frac{f(x_0+h)-f(x_0)}{x}$

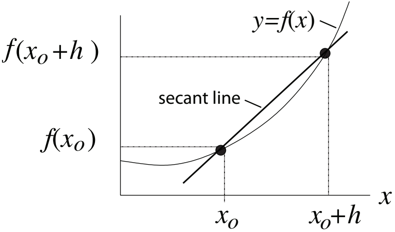
E. $\frac{f(x_0+h)-f(x_0)}{h}$



Average rate of change

The average rate of change of $y = f(x)$ over the interval $x_0 \leq x \leq x_0 + h$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$



Zebrafish development

https://en.wikipedia.org/wiki/Zebrafish

- Posterior lateral line primordium migration: https://youtu.be/IqUs29Kz3HE

Zebrafish development

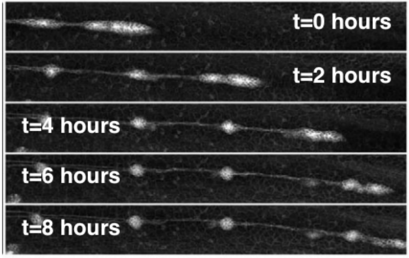


Figure modified from Valdivia et al. *Development* 2011.

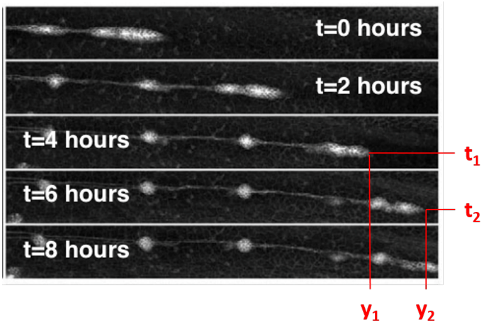
Q3. Over time the cell cluster is

A. speeding up

B. slowing down

C. moving at the same speed

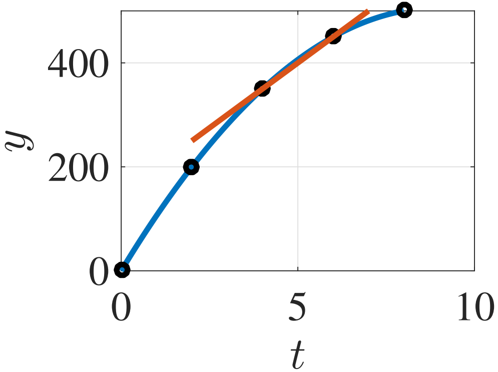
Average velocity



Suppose the cell cluster is at position y_1 at t_1 and at y_2 at t_2. The average velocity over the interval $t_1 \leq t \leq t_2$ is

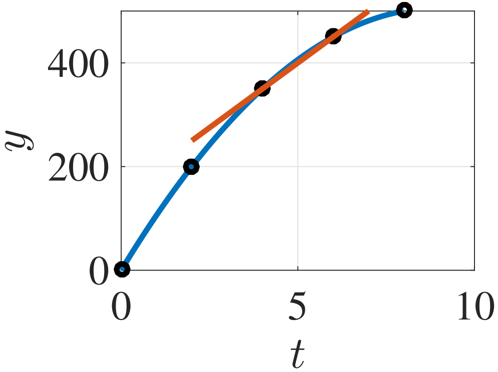
$$v_{\text{average}} = \frac{\text{distance traveled}}{\text{time taken}} = \frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1}$$

Zebrafish development



- Blue curve: position as a function of time
- Black dots: measured data points
- Red line: secant line through (t_1, y_1) and (t_2, y_2)

Zebrafish development



$$v_{\text{average}} = \frac{\Delta y}{\Delta t} = \frac{y_2 - y_1}{t_2 - t_1}$$

which is the slope of the secant line through (t_1, y_1) and (t_2, y_2).

Zebrafish development

time t (hours)	position y (μm)
0	0
2	200
4	350
6	450
8	500

Q4. Over $2 \leq t \leq 4$ hours, the average velocity of the cell cluster was (spreadsheet time permitting)

A. 150 μm/h

B. 100 μm/h

C. 75 μm/h

D. 50 μm/h

E. 25 μm/h

Zebrafish development

time t (hours)	position y (μm)
0	0
2	200
4	350
6	450
8	500

Q5. Over $6 \leq t \leq 8$ hours, the average velocity of the cell cluster was (spreadsheet time permitting)

A. 150 μm/h

B. 100 μm/h

C. 75 μm/h

D. 50 μm/h

E. 25 μm/h

Instantaneous rate of change?

A way to define/compute the Instantaneous rate of change?

Limits

What is the value of $10 + x$ when x is very close to 0 ($|x| \ll 10$)?

$$10 + x \approx 10$$

$$\lim_{x \to 0} 10 + x = 10$$

Fact (Limits of polynomials)

A polynomial $f(x)$ is a continuous function. Therefore

$$\lim_{x \to a} f(x) = f(a).$$

Limits

Q6. Compute the limit

$$\lim_{x \to 1} x^3 - 2x^2 - 1$$

A. 1
B. -2
C. 0
D. -1
E. Does not exist (DNE)

Limits

Q7. Compute the limit

$$\lim_{x \to 2} \frac{x+1}{x-2}$$

A. 1

B. 2

C. 0

D. -2

E. Does not exist (DNE)

- This function "blows up" as $x \to 2$ (division by zero), so the limit DNE.

Limits

Q8. Compute the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

A. 4
B. 2
C. 0
D. -4
E. Does not exist (DNE)

Limits

$$\frac{x^2-4}{x-2} = \begin{cases} x+2, & \text{if } x \neq 2 \\ \text{undefined}, & \text{if } x = 2 \end{cases}$$

The function is not defined at $x = 2$ but it does have a limit there:
https://www.desmos.com/calculator/hoofsji5p4

$$\begin{aligned} \lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2}(x+2) \\ &= 4 \end{aligned}$$

Instantaneous rate of change at x_0

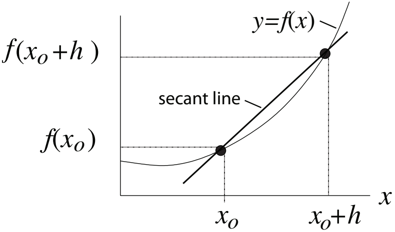
- Average rate of change:

$$\frac{f(x_0+h)-f(x_0)}{h}$$

- Take limit as $h \to 0$:

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

- defined as instantaneous rate of change.



Falling object

- Galileo (about 400 years ago): distance fallen versus time:

$$y(t) = ct^2$$

(with $c = 4.9$ m/s^2).

Q9. Compute the average velocity $\bar{v}$ over the time interval $t_0 \leq t \leq t_0 + h$.

A. $2ct_0 + h$

B. $2ct_0$

C. $c(2t_0 + h)$

D. $2ct$

E. Not sure

Falling object

Function: $y(t) = ct^2$, interval: $t_0 \leq t \leq t_0 + h$.

$$\begin{aligned} v_{\text{average}} &= \frac{y(t_0+h) - y(t_0)}{h} \\ &= \frac{c(t_0+h)^2 - c(t_0)^2}{h} \\ &= c\left(\frac{(t_0^2 + 2ht_0 + h^2) - (t_0)^2}{h}\right) \\ &= c\left(\frac{2ht_0 + h^2}{h}\right) \\ &= c(2t_0 + h) \end{aligned}$$

Falling object

Q10. What happens to the average velocity, $c(2t_0 + h)$, as $h \to 0$?

A. $2c$

B. ch

C. $2ch$

D. $2ct_0$

E. Not sure

Falling object

The instantaneous velocity at time t_0 is

$$v(t_0) = \lim_{h \to 0} \frac{y(t_0 + h) - y(t_0)}{h},$$

which is also called the derivative of the function $y(t)$ at time t_0.

Derivative

Definition (Derivative)

The derivative of a function $y = f(x)$ at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We can also write $\left.\frac{df}{dx}\right|_{x_0}$ to denote $f'(x_0)$

- The derivative depends on x_0
- It gives the rate of change of $f(x)$ at the instant that $x = x_0$
- It is obtained as a formal algebraic process involving a limit

Derivative: Geometric view

- As $h \to 0$, the secant line approaches a tangent line
- The slope of the tangent line at the point x is the derivative of the function at the given point x
- Visualisation:
  https://www.desmos.com/calculator/sddxz6kzbp
- Talk about equation for tangent lines (time permitting)

Today...

- Average rate of change and secant lines
- Limits
  - plug-in for continuous functions
  - factoring
  - Some limits DNE
- Instantaneous rate of change: take $h \to 0$ in the average rate of change
- The derivative of $y = f(x)$ at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- As $h \to 0$, the secant line approaches a tangent line

Answers

1. B
2. E
3. B
4. C
5. E
6. B
7. E
8. A
9. C
10. D

Related exam problems

1. Which of the following describes the derivative of a function $f(x)$?
   A. It is the slope of the secant line on the graph of $f(x)$.
   B. It is the average rate of change of $f(x)$ over the interval $0 < x < h$.
   C. It is defined as $\frac{f(x+h)-f(x)}{h}$.
   D. More than one of the above answers are correct.
   E. None of the above are correct.