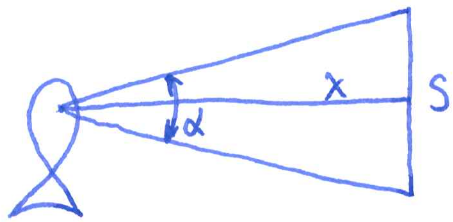
Escape response



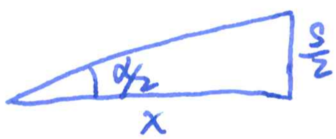
α = visual angle

S = predator size (constant)

x = distance to predator

know: $\frac{dx}{dt} = -v$ (constant)

1. want: $\frac{d\alpha}{dt} = ?$ (a function of s, v, x)



$$\tan\left(\frac{\alpha}{2}\right) = \frac{\frac{s}{2}}{x} = \frac{s}{2x}$$

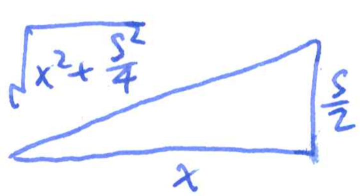
To remind ourselves: $\tan\left(\frac{\alpha(t)}{2}\right) = \frac{s}{2x(t)}$

Differentiate w.r.t. t:

$$\frac{1}{\cos^2\left(\frac{\alpha(t)}{2}\right)} \cdot \frac{\alpha'}{2} = \frac{-s}{2x^2(t)}\, x'$$

$$\frac{d\alpha}{dt} = -\frac{s}{x^2(t)} \cos^2\left(\frac{\alpha(t)}{2}\right) \frac{dx}{dt} = \frac{sv}{x^2} \underbrace{\cos^2\left(\frac{\alpha}{2}\right)}$$

what's this in terms of s, x?



$$\cos\left(\frac{\alpha}{2}\right) = \frac{x}{\sqrt{x^2+\frac{s^2}{4}}}$$

So $$\frac{d\alpha}{dt} = \frac{sv}{x^2} \cdot \frac{x^2}{x^2+\frac{s^2}{4}} = \frac{sv}{x^2+\frac{s^2}{4}}$$

According to our theory, escape is triggered when $\frac{d\alpha}{dt}$ is sufficiently large. Let's say, when

$$\frac{d\alpha}{dt} > \omega_{crit}.$$

2. Investigate different variables/parameters affecting $\frac{d\alpha}{dt}$.

① The smaller x is, the closer a predator is, and the larger $\frac{d\alpha}{dt}$ is (because of the x^2 in the denominator). So escape may be more easily triggered if the predator is closer.

② $\frac{d\alpha}{dt} = \frac{s}{x^2+\frac{s^2}{4}} \cdot v \propto v$. Hence, the faster the predator moves, the faster it appears in the zebrafish's eyes. Escape is more likely to happen with a big v.

③ The effect of the parameter s, however, is not obvious. Let's sketch the function

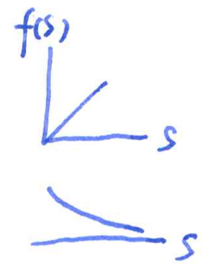
$$f(s) = \frac{sv}{x^2 + \frac{s^2}{4}}$$

(Remember curve sketching?)

(i) Qualitative behavior

$s \to 0$, $f(s) \approx \frac{sv}{x^2} \propto s$

$s \to \infty$, $f(s) \approx \frac{sv}{\frac{s^2}{4}} = \frac{4v}{s}$



(ii) Roots

$$\frac{sv}{x^2+\frac{s^2}{4}}=0 \Rightarrow s=0$$

(iii) CPs

$$f'(s) = \frac{v(x^2+\frac{s^2}{4}) - sv(\frac{2s}{4})}{(x^2+\frac{s^2}{4})^2} = \frac{vx^2+\frac{s^2v}{4}-\frac{s^2v}{2}}{(x^2+\frac{s^2}{4})^2}$$

quotient rule

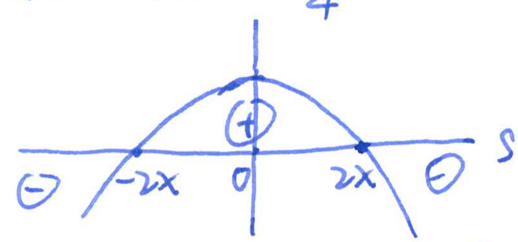
$$= \frac{vx^2-\frac{s^2v}{4}}{(x^2+\frac{s^2}{4})^2}$$

$$f'(s)=0 \Rightarrow vx^2=\frac{s^2v}{4}, \quad s^2=4x^2, \quad s=\pm 2x$$

(Don't forget to classify the CPs. They can be a max, a min, or an IP. Use either FDT or SDT to classify an extremum.)

Notice $(x^2+\frac{s^2}{4})^2>0$. Then $f'(s)$ has the same sign with $vx^2-\frac{s^2v}{4}$. Use FDT:

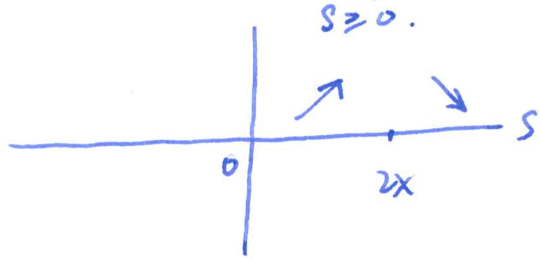
$f'(s)$:



remember we are considering ~~a function~~ $vx^2-\frac{s^2v}{4}$ as a function of s, not x!

only need $s\geq 0$.

$f(s)$:



Monotonicity characterized by FDT

(iv) IPs and concavity.

$$f''(s) = \frac{-\frac{2sv}{4}\left(x^2+\frac{s^2}{4}\right)^2 - \left(vx^2-\frac{s^2v}{4}\right)\cdot 2\left(x^2+\frac{s^2}{4}\right)\cdot\frac{2s}{4}}{\left(x^2+\frac{s^2}{4}\right)^4}$$

$$= \frac{-\frac{sv}{2}\left(x^2+\frac{s^2}{4}\right) - s\left(vx^2-\frac{s^2v}{4}\right)}{\left(x^2+\frac{s^2}{4}\right)^3}$$

$$= \frac{-\frac{svx^2}{2} - \frac{s^3v}{8} - svx^2 + \frac{s^3v}{4}}{\left(x^2+\frac{s^2}{4}\right)^3}$$

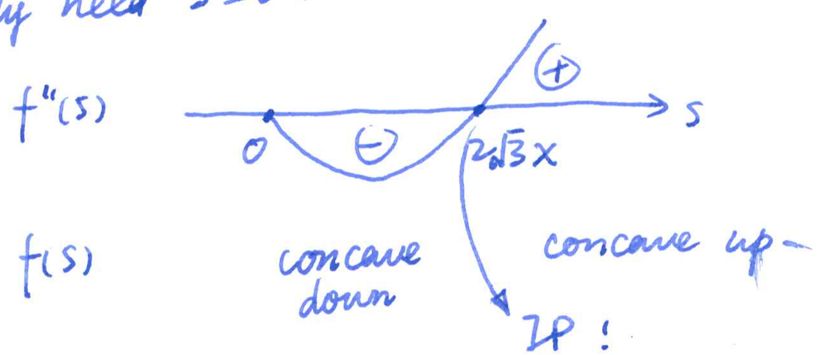
$$= \frac{-\frac{3}{2}svx^2 + \frac{1}{8}s^3v}{\left(x^2+\frac{s^2}{4}\right)^3}$$

$$= \frac{sv(s^2-12x^2)}{8\left(x^2+\frac{s^2}{4}\right)^3}$$

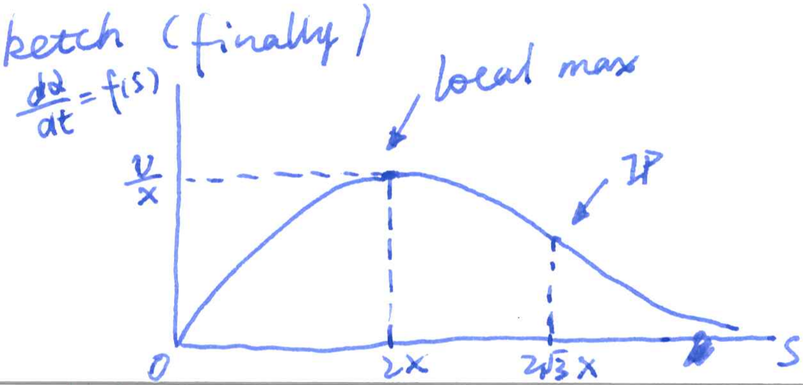
$f''(s)=0 \Rightarrow sv(s^2-12x^2)=0, \quad sv(s-2\sqrt{3}x)(s+2\sqrt{3}x)=0$

$s = 0, \pm 2\sqrt{3}x.$

Only need $s \geq 0$:



(v) Sketch (finally)



!! Surprising.
When the predator is large, it appears to be moving slowly in the eyes of the zebrafish! The larger it is, the slower it appears!
— Dangerous.

3. We can look at the model from another perspective. Thinking about a predator coming close with velocity v, Let's calculate at what distance x_{react} the zebrafish's escape response is triggered?

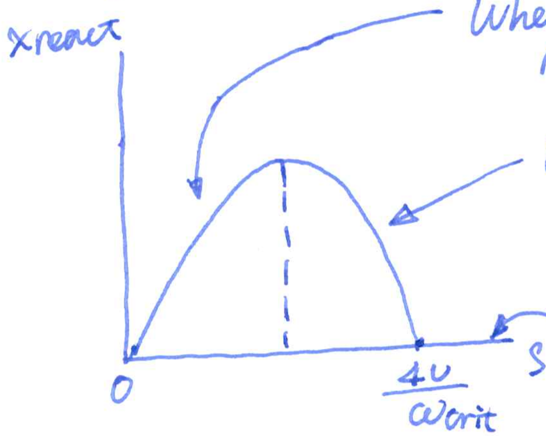
Escape starts when $\frac{d\alpha}{dt} = \omega_{crit}$. So

$$\frac{sv}{x_{react}^2 + \frac{s^2}{4}} = \omega_{crit}$$

$$x_{react}^2 + \frac{s^2}{4} = \frac{sv}{\omega_{crit}}$$

$$x_{react} = \sqrt{\frac{sv}{\omega_{crit}} - \frac{s^2}{4}} = \sqrt{s\left(\frac{v}{\omega_{crit}} - \frac{s}{4}\right)}$$

If we sketch x_{react} as a function of s:
(details omitted)



When $s\uparrow$, $x_{react}\uparrow$, giving more room for escape. Good!

When s is large, because the predator appears slow as we have found out just now, there's less room to react!

If $s > \frac{4v}{\omega_{crit}}$, i.e. $s\omega_{crit} > 4v$, the zebrafish has not room to react. It does not escape at all.

So a predator that is big and slow is the most dangerous.

(One more message next page.)

Thank you all for taking this class, especially at 8:00 am. It's been a pleasure working with all of you. With all your hard work, I hope you have learned something from this challenging course. Math, as the language for science, can give us a whole new perspective and insights, even for a traditionally experimental subject like biology. I hope you can keep exploring!

If you want to share thoughts, feel free to send me an email, any time now or in the future. I hope you have been enjoying college. I wish you every success in life.

Merry Christmas!

Sincerely,
Mingfeng