Derivatives of Trig and Inverse Trig Functions

Math 102 Section 102
Mingfeng Qiu

Nov. 28, 2018

Office hours

I'm planning to have additional office hours next week. Next Monday (Dec 3), which time works best for you?

A. 1:00-2:00 (most popular)

B. 2:00-3:00 (next popular)

C. 4:00-5:00

D. 5:00-6:00

Last time

- Trigonometric functions can be used to describe rhythmic processes. To do that, one just needs to figure out the amplitude, frequency and phase shift.
- To be invertible, trig functions have to be restricted to certain domains.
- Triangles and identities are useful when simplifying inverse trigonometric relationships.

Today: learning goals

- Define and calculate derivatives of trig functions
- and those of their inverses
- Use trig functions in related rates problems

Derivatives of trig functions

Derivative of cosine and sine

Cosine:

$$\frac{d}{dt}\cos t = -\sin t$$

Sine:

$$\frac{d}{dt}\sin t = \cos t$$

The special relationship between these functions means that they satisfy the following equations:

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = x$$

Derivative of cosine and sine: visualization

- Recall: the derivative of displacement in time = velocity
- Recall: trig functions are closely related to motion along a circle
- Desmos demo

Derivative of $\sin x$

$$
\begin{aligned}
\frac{d\sin(x)}{dx} &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right)+\cos(x)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right) \\
&= \cos(x)
\end{aligned}
$$

Second derivative of $y(t) = \sin(t)$?

Q1. $y = \sin(t)$ is a solution to the following differential equation.

A. $\frac{d^2y}{dt^2} = -y$

B. $\frac{d^2y}{dt^2} = y$

C. $\frac{dy^2}{dt} = y$

D. $\frac{d^2y}{dt^2} = -t$

E. Send help

Second derivatives

$$\frac{d^2 \sin t}{dt^2} = -\sin t \Rightarrow \frac{d^2 y}{dt^2} = -y$$

$$\frac{d^2 \cos t}{dt^2} = -\cos t \Rightarrow \frac{d^2 x}{dt^2} = -x$$

- The trig functions $\sin t$ and $\cos t$ are both solutions to the differential equation

$$\frac{d^2 y}{dt^2} = -y$$

Derivative of $\tan x$

Q2. The derivative of $\tan x$ is

A. $\cot x$

B. $-\cot x$

C. $\cos^2(x)$

D. $\sec^2(x)$

E. Send help

Use quotient rule.

Derivatives of inverse trig functions

Derivatives of inverse trig functions

Q3. If $y = \arcsin x$, with $-1 \leq x \leq 1$, what is $\frac{dy}{dx}$?

A. $\arccos x$

B. $\arctan x$

C. $\sqrt{1-x^2}$

D. $\frac{1}{\sqrt{1-x^2}}$

E. $\frac{1}{1+x^2}$

In order to solve this problem, it's actually easier to think about the derivative of a general inverse function $f^{-1}(x)$.

And how did we find the derivative of a log function?

Derivatives of inverse functions

- Consider an inverse function $y = f^{-1}(x)$
- This means $x = f(y)$
- Take derivative w.r.t. x on both sides!

$$1 = f'(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

- The derivative of the inverse function is the reciprocal of the derivative of the original function! Be careful of variables. Don't forget to substitute y with the appropriate expression in x!

Derivatives of Inverse Trig Functions

$y = f(x)$	$f'(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{x^2+1}.$

Trig functions and related rates

Search light in a prison

Example

In a prison, a search light is 30 m from a straight wall. The search light rotates at a rate of 3 rounds per minute. Let P be the point on the wall closest to the light tower. The light beam hits the wall at some spot (with negligible cross-sectional width), and that spot moves as the search light rotates. When the light spot on the wall is 10 m away from P, how fast is the spot moving?

(Document camera)

Summary

- $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$
- These derivatives can be demotrated with motion on a circle, and derived rigorously using the limit definition.
- Derivatives of other trig functions may be found by using differentiation rules.
- Derivatives of inverse trig functions can be derived with implicit differentiation.

Extra Practice

1. Show that the derivative of $\arctan x$ is $\frac{1}{1+x^2}$

Answers

1. C
2. D
3. D

Related Exam Problems

1. Use Newton's method to find the smallest positive critical point of the function

$$g(x) = e^{-x}\sin(10x)$$