Disease Dynamics

Use a "mass" balance relationship:

We track the # of infected people $I(t)$

rate of change of I =

rate of infection − rate of recovery

use law of mass action. $= \mu I$

$= \beta S I$

$$\Rightarrow \frac{dI}{dt} = \beta S I - \mu I$$

What if tracking $S(t)$?

$$\frac{dS}{dt} = -\frac{dI}{dt} = -\beta S I + \mu I$$

Another way of seeing this:

$S + I = N$, constant

$S = N - I \Rightarrow \frac{dS}{dt} = -\frac{dI}{dt}$

Simplify:

$$\frac{dI}{dt} = \beta I(N-I) - \mu I$$

$$= \beta I\Big(\underline{N - \frac{\mu}{\beta}} - I\Big)$$

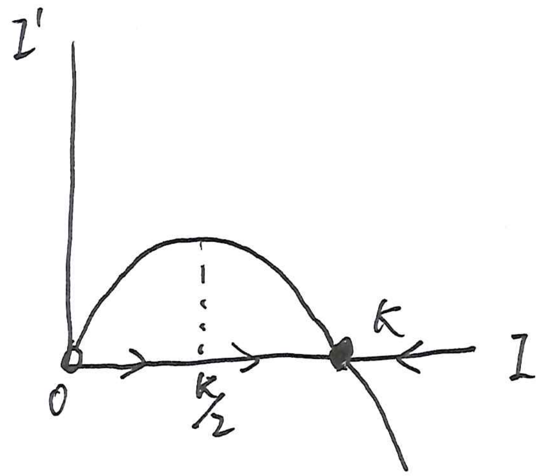
Let $K = N - \frac{\mu}{\beta}$ $\Rightarrow$ $\underline{\frac{dI}{dt} = \beta I(K-I)}$.

Qualitative analysis

SS: set $\frac{dI}{dt} = 0 \Rightarrow \beta I(K-I) = 0 \Rightarrow I = 0, K.$

① If $K > 0$, i.e., $N - \frac{\mu}{\beta} > 0 \Leftrightarrow N\beta > \mu$

$$\frac{dI}{dt} = \beta I(K-I) \rightarrow \text{logistic eqn in disguise.}$$

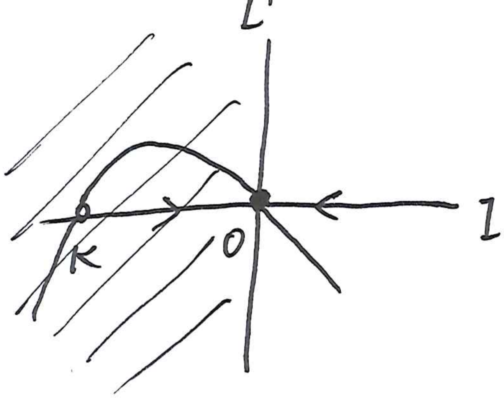


$$\frac{dN}{dt} = rN\Big(1 - \frac{N}{K}\Big) = \Big(\frac{r}{K}\Big)N(K-N)$$

As long as $I(0) > 0$, $I(t) \rightarrow K$ as $t \rightarrow \infty$.

② If $K<0$, i.e., $N\beta<\mu$

$I=K$ is not admissible.



$I(t)\to 0$, $t\to\infty$.

Interpretation:

① $K>0$. The disease becomes endemic. There are always a group of sick people of size K.

② $K<0$. The disease is wiped out.

The model captures the phenomena through $K=N-\frac{\mu}{\beta}$. (or the basic reproduction number $R_0=\frac{N\beta}{\mu}$, $K>0 \Leftrightarrow R_0>1$.) It all gets down to the relative magnitude of N, β, μ.