Solution

Worksheet: Euler's method
Math 102 Section 102
Nov. 16, 2018

Use a computer. Work in a group.

(Exponential growth) Consider the exponential growth model, with per-capita growth rate $r = 1$ and initial condition $N_0 = N(0) = 1$:

$$\frac{dN}{dt} = N, \quad N(0) = 1.$$

which has solution $N(t) = e^t$

1. In the context of Euler's method, what is $f(N)$ for this model?

   $f(N) = N$

2. Let h be the step size. Determine the Euler iteration formula for the exponential growth model:

   $N_{n+1} = N_n + h \cdot f(N_n) = (1+h) N_n$

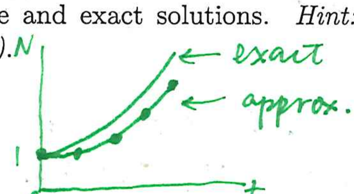
3. Suppose we want to approximate the solution to the exponential growth model from $t = 0$ to $t = 1$ using 5 steps. What is the step size h?

   $h = \frac{1-0}{5} = 0.2$

4. Fill in the following table. (see the link to Google sheets in the lecture slides)

Time t	Counter n	Approximate N_n	Exact $N(t) = e^t$	Error $\lvert N_n - N(t) \rvert$
0	0	1	1	0
0.2	1	1.2	1.2214	0.0214
0.4	2	1.44	1.4918	0.0518
0.6	3	1.728	1.8221	0.0941
0.8	4	2.0736	2.2255	0.1519
1	5	2.4883	2.7183	0.2300

5. Plot the approximate and exact solutions. *Hint: use a line chart (with markers for the approximate solution).*



6. What is the maximum error with 5 time steps? *Hint: use max(cell:range).* Repeat the calculation with 10, 20 and 40 time-steps. What happens to the maximum error as the time-step is doubled?

# of time steps	max error
5	0.2300
10	0.1245
20	0.0650
40	0.0332

As the # of time steps doubles, the max error decreases approximately by a factor of 2.

(In fact, the error behaves as $\lvert N_n - N(t) \rvert = C(t) h$)

How to get this? You will know after learning about Taylor expansion in your future courses. The field specialized in studying numerical sol'ns is called "numerical analysis"

(Logistic growth and stability of Euler's method) The logistic equation,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

has solution

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}.$$

(solving the logistic equation is beyond the scope of Math 102).

1. With $r = 0.1$ and $K = 1000$, what is $f(N)$ in the context of Euler's Method?

   $f(N) = rN\left(1 - \frac{N}{K}\right) = 0.1\,N\left(1 - \frac{N}{1000}\right)$

2. Determine the Euler iteration formula for the exponential growth model:

   $$N_{n+1} = N_n + h \cdot f(N_n) = N_n + 0.1\,h\,N_n\left(1 - \frac{N_n}{1000}\right)$$

3. $N_0 = 2$. Use a timestep of $h = 25$ to find an approximate solution up to time $t = 200$. What happens? (called "unstable")

   The approximate sol'n oscillates and does not match the exact!

4. Use a timestep of $h = 1$ to find an approximate solution up to time $t = 200$. What happens?

   The approximate sol'n is close to the exact sol'n.

5. What is the disadvantage of choosing a small timestep?

   One has to use a lot of time steps.

   (One can design more sophisticated numerical schemes that both are stable and can use large time step sizes. The cost, however, is more calculation to do at each time step.)

(Stability) Use Euler's Method to find a solution to

$$\frac{dy}{dt} = -3y, \quad y(0) = 1$$

with step size $h = 1$.

1. What's the analytical solution to this Initial Value Problem (IVP)? Does the solution exponentially grow or decay?

   $y(t) = e^{-3t}$. It exponentially decays.

2. What happens to the approximate solution over time? Does the approximate solution exhibit behavior consistent with your prediction above?

   The approximate sol'n oscillates and grows in time! (unstable)
   — inconsistent with our expectation $\Rightarrow$ large error.

3. What timestep size, h, should you use to ensure that the solution exhibits the correct qualitative behaviour?

   We need a small time step. (Choose an "h" such that $h < \frac{2}{3}$)

   No need to know where this number comes from. Just try some small h values.

Learn more. Euler's Method produces good approximations to solutions to some equations, provided that the timestep h is small enough. You can learn more about Euler's Method and related *Numerical Methods* at `https://en.wikipedia.org/wiki/Euler_method`.

If you are curious how we figure out all this error/stability business, consider taking a course in numerical analysis later.