Numerical Solution of DEs: Euler's Method

Math 102 Section 102
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A WebWork question

Assignment 10: Problem 1

Previous Problem | Problem List | Next Problem

(1 point) **MATH102HWProblems/solving_differential_equations/Cubical_crystal**

A crystal grows inside a medium in a cubical shape with side length x and volume V. The rate of change of the volume is given by

$$\frac{dV}{dt} = kx^2(I - V)$$

where k and I are positive constants.

(a) Rewrite this as a differential equation for $\frac{dx}{dt}$.

$\frac{dx}{dt} =$ ______

(b) What happens to the size of the crystal after a very long time?

$x \to$ ______

Apparently you don't know how to solve this equation. What can you do? What did we learn recently?
The question is asking for qualitative behavior of x when t goes to infinity.
Can we tell that without solving the equation analytically?

Today: learning goals

1. An example for Newton's Law of Cooling
2. Euler's method: concept, algorithm
3. Implementing Euler's method using a worksheet

Newton's law of cooling (or heating)

The rate of change of temperature T of an object is proportional to the difference between its temperature and the ambient temperature, E.

$$\frac{dT}{dt} = -\alpha \left(T(t) - E \right).$$

Q-last time. If $T(0) = T_0$, then

A. $T(t) = E + (T_0 - E)e^{\alpha t}$.

B. $T(t) = E + (E - T_0)e^{\alpha t}$.

C. $T(t) = E + (T_0 - E)e^{-\alpha t}$.

D. $T(t) = E + (E - T_0)e^{-\alpha t}$.

$$\frac{dT}{dt} = \alpha E - \alpha T.$$

Let $a = \alpha E,\ b = \alpha.$ Use the analytical solution for the equation like $\frac{dy}{dt} = a - by$.

Newton's law of cooling (or heating)

"Under certain circumstances there are few hours in life more agreeable than the hour dedicated to the ceremony known as afternoon tea."
—Henry James (1843-1916), *The Portrait of a Lady*

Newton's law of cooling (or heating)

Example (Tea)

A cup of just-boiled tea is put on a table. After ten minutes, the tea is 40°,and after 20 minutes, the tea is 25°. What is the ambient temperature? (Document camera)

Numerical solution of DEs

- Analytic solution to a DE: explicit formula for the solution
  - Also called exact solution
- Numerical solution to a DE: approximation to the exact solution using a procedure
  - The procedure is also called an algorithm or numerical scheme
- A numerical solution is an approximation!
  - Error induced by your algorithm
  - Error caused by your computer

Euler's Method

- Euler's Method (pronounced Oil-err) is a method to numerically approximate the solution to a differential equation

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0$$

(can be extended to non-autonomous DEs)

Euler's Method

Q1. True or false?

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h} \quad \text{when } h \text{ is small}$$

A. True
B. False

It's an approximation to the derivative using the slope of a secant line. Also called finite difference approximation to the derivative.

Euler's Method

$$f(y) = \frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h}$$

Assume that in the LHS, take $f(y(t))$

- Taking y at different time points will end up with different numerical schemes.

Q2. Which of the following is correct?

A. $y(t+h) = y(t) - hf(y(t))$

B. $y(t+h) = y(t) + hf(y(t))$

C. $y(t+h) = y(t) - hf(y(t+h))$

D. $y(t+h) = y(t) + hf(y(t+h))$

Euler's Method

$$\frac{dy}{dt} = f(y) \quad \Rightarrow \quad y(t+h) = y(t) + hf(y(t))$$

Q3. Suppose you know the solution value y_n at some point in time t_n. Following the above formula, to get an approximate value of y at the next time point $t_{n+1} = t_n + h$, we should make

A. $y_{n+1} = y_n - hf(y_n)$

B. $y_{n+1} = y_n + hf(y_n)$

C. $y_{n+1} = y_n - hf(y_{n+1})$

D. $y_{n+1} = y_n + hf(y_{n+1})$

Euler's Method

To approximate the solution to

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0$$

- Choose a time step size h, and divide the time axis into small steps each with length h.
- At each step t_{n+1}, generate an approximation using Euler's Method:

$$y_{n+1} = y_n + hf(y_n), \quad \text{with } y_0 = y(0)$$

(Idea: making small discrete steps in time. At each step, use a linear approximation to get the next point.)

Euler's Method Demo and Worksheet

Google sheets link

Related Exam Problems

1. Describe how you know whether or not Euler's Method gives an over or underestimate of the solution?