Applications of Qualitative Analysis and Solving Diffetrential Equations

Math 102 Section 102
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Nov. 14, 2018

Due due due due due...

- Nov 14 (Today): Pre-lecture 11.2
- Nov 15 (Tomorrow): Assignment 10

Assignments due: 9:00 pm

Annoucements

- Today: last day for midterm regrade requests
- Final exam: Dec. 4th, 3:30 pm @SRC (Info page on Canvas)
- Tools for slope fields and state-space diagrams
  - Desmos slope field generator
  - dfield and pplane
- Did you work on the worksheet from last Wed?
  - A. Yes! I worked on it, and looked at the solution.
  - B. Yes! I worked on it, but did not look at the solution.
  - C. No! But I studied the solution.
  - D. No! Nor did I study the solution.
  - E. What worksheet? What are you talking about?

Last time

Solve the Initial Value Problem (IVP) involving a first-order linear DE

$$\frac{dy}{dt} = a - by,$$

such that

$$y(0) = y_0.$$

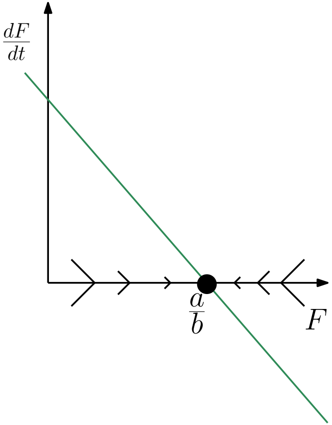
- Rewrite the DE and IC in terms of deviation from the steady state (transform of variables) ($z = y - a/b$)
- Solve the simplified equation ($dz/dt = -bz \;\Rightarrow\; z = Ce^{-bt}$)
- Convert the variable back to the original ($y = a/b + (y_0 - a/b)e^{-bt}$)

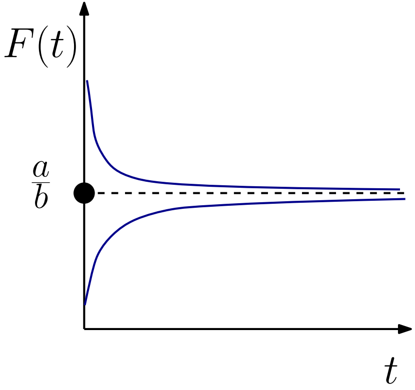
Today: learning goals

- Apply qualitative analysis and the analytical solution to solve application problems
- Interpret results in specific application contexts
  - Our data shows: typical difficulty in exams!

Qualitative analysis of a simple DE

$$\frac{dF}{dt} = a - bF.$$ (Document camera)





Qualitative analysis of a simple DE

Does it agree with our analytical solution?

$$F(t) = \frac{a}{b} + \left(F_0 - \frac{a}{b}\right) e^{-bt}.$$

Yes! $F(t)$ exponentially decays/grows towards $\frac{a}{b}$, regardless of F_0.

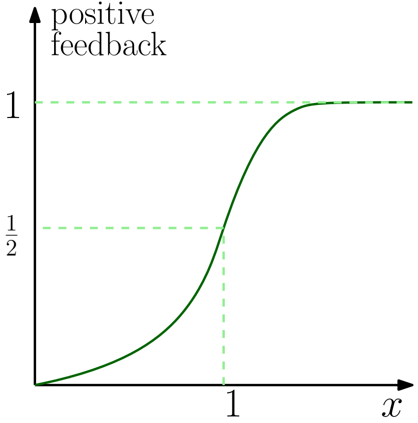
A biological switch

- In many cells, a gene may turn "on" due to a signal (from the environment, for example) and produce a protein X,
- but the protein will decay at a rate proportional to its concentration (the gene turns "off" after some time).
- If the protein level is sufficiently high, there may be additional positive feedback, which keeps the gene "on" for longer, and boosts the protein production.
- The concentration of the protein satisfies:

$$\frac{dx}{dt} = f(x) = \frac{x^2}{1+x^2} - mx$$

A biological switch

$$\underbrace{\frac{dx}{dt}}_{\text{rate of change}} = \underbrace{\frac{x^2}{1+x^2}}_{\text{positive feedback}} - \underbrace{mx}_{\text{decay}}$$



- When $x > 1$, the rate of production of x is high
- When $x < 1$, the rate of production of x is low

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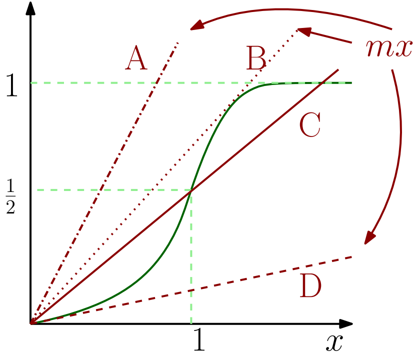

- Depending on m, the line mx has a different slope
- We will focus on C together, and you can figure out the remaining cases

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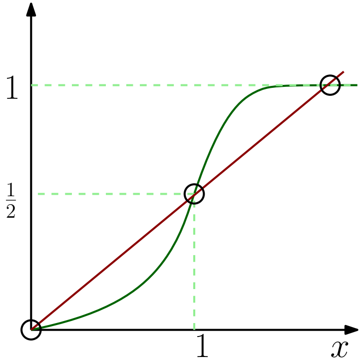

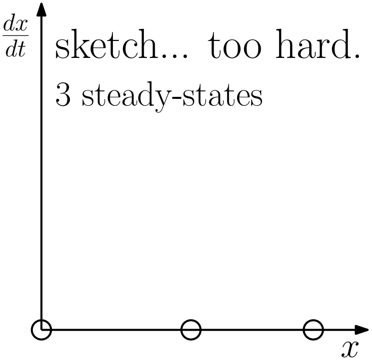

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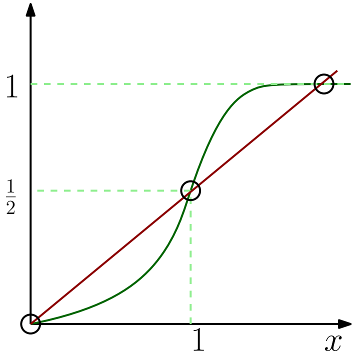

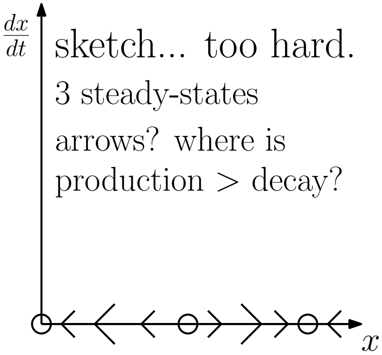

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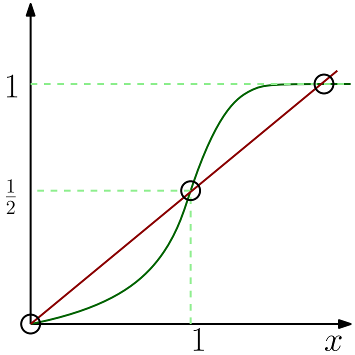

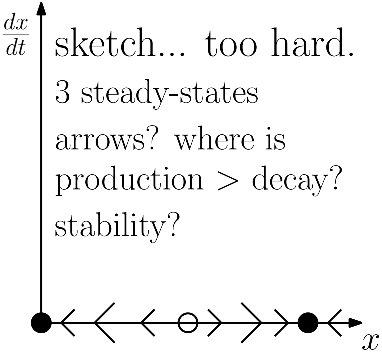

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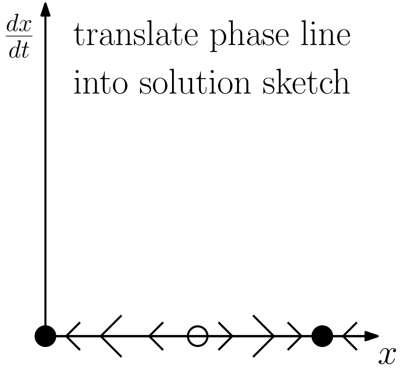

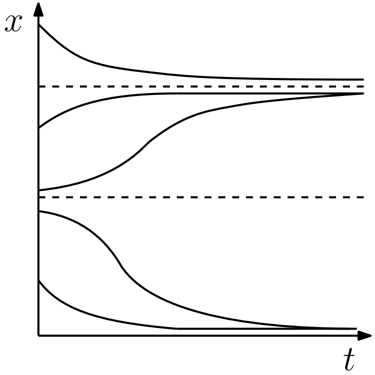

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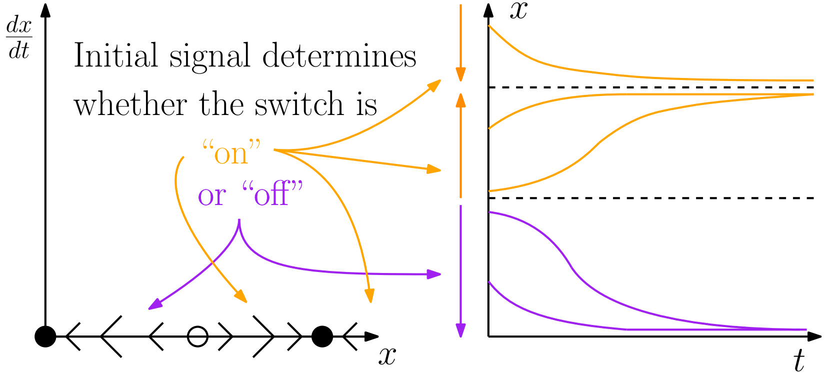

Newton's law of cooling (or heating)

The rate of change of temperature T of an object is proportional to the difference between its temperature and the ambient temperature, E.

$$\frac{dT}{dt} = k\left(T(t) - E\right).$$

Q1. Is k positive or negative?

A. Positive.

B. Negative.

C. It depends on the situation.

Newton's law of cooling (or heating)

The rate of change of temperature T of an object is proportional to the difference between its temperature and the ambient temperature, E.

$$\frac{dT}{dt} = -\alpha \left(T(t) - E \right).$$

Q2. If $T(0) = T_0$, then

1. $T(t) = E + (T_0 - E)e^{\alpha t}$.
2. $T(t) = E + (E - T_0)e^{\alpha t}$.
3. $T(t) = E + (T_0 - E)e^{-\alpha t}$.
4. $T(t) = E + (E - T_0)e^{-\alpha t}$.

$$\frac{dT}{dt} = \alpha E - \alpha T.$$

Let $a = \alpha E,\ b = \alpha$. Use the analytical solution for the equation like $\frac{dy}{dt} = a - by$.

Summary

- We can use qualitative methods to understand model behaviour when we cannot solve the differential equation
- We can solve some differential equations ($y' = a - by$, $y(0) = y_0$)
- Equations of this type arise in many scientific examples (e.g., Newton's Law of Cooling)

Answers

1. B
2. C

Related Exam Problems

1. Consider Newton's Law of cooling

$$\frac{dT}{dt} = 2 - \frac{1}{5}T$$

with initial condition $T(0) = 37$.

(a) Find values of the constants a,b and k such that

$$T(t) = a + be^{-kt}$$

is a solution to the initial value problem given above.

(b) Using the solution obtained in (a) to find the time τ at which $T(\tau) = 13$. Express the answer in terms of m, where $m = \ln(3)$.

(c) What is the steady state for this differential equation? Is it stable or unstable?

Related Exam Problems

2. The temperature of a cup of coffee is initially 100 degrees C. Five minutes later, it is 50 degrees C. If the ambient temperature is $A = 20$ degrees C, determine how long it takes for the temperature of the coffee to reach 30 degrees C.