Solving differential equations of the form $\frac{dy}{dt} = a - by$.

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You already know:
- Why we care about diff. equations
- What the solution is for a diff. equation of the form $\frac{dy}{dt} = ky$
- How to graphically interpret diff. equations using slope fields and state space diagrams.

Today we will see: How to use what we already know to find solutions for diff. equations of the form $\frac{dy}{dt} = a - by$ $(y(0) = y_0)$.

Part 1

Special solutions: steady states.

We have $\frac{dy}{dt} = a - by$

Question: are there values of y for which $\frac{dy}{dt} = 0$?

This would mean that

$$a - by = 0$$
$$by = a$$
$$y = \frac{a}{b}.$$

Solution: $y(t) = \frac{a}{b}$.

These constant solutions are called <u>steady states</u>.

Part 2 $a=0$.

Consider case $a=0$. So $\frac{dy}{dt}=-by$, $y(0)=y_0$

We know ~~how~~ what the solution to this is!

Solution: $y(t)=Ce^{-bt}$

Use initial value: $y_0=y(0)=C$

So $y(t)=y_0e^{-bt}$.

Part 3 How can we generalise this to $a\neq 0$?

We have $\frac{dy}{dt}=a-by$, $y(0)=y_0$ ⊛

$$=-b\underbrace{\left(y-\frac{a}{b}\right)}$$

This is the difference or <u>deviation</u> of y away from its steady state value. ~~as a function of time~~

Now we reformulate the DE. into one whose solution we know.

Let $z(t)=y(t)-\frac{a}{b}$. i.e. z is the deviation of y from its steady state value as a function of time.

We want to convert ⊛ into a DE in terms of z.

We need to deal with 3 things:

- LHS
- RHS
- Initial value.

RHS $-b\left(y-\frac{a}{b}\right)=-bz$

LHS $\frac{dz}{dt}=\frac{dy}{dt}$

IV $z(0)=y(0)-\frac{a}{b}=y_0-\frac{a}{b}$.

We know the solution to this!

We get $z(t) = Ce^{-bt}$

Use IV: $z(0) = y_0 - \frac{a}{b} = C$

Solution to (**): $z(t) = (y_0 - \frac{a}{b})e^{-bt}$.

Are we done? Not quite!

We want to convert this back into a solution for (*).

We have $z = y - \frac{a}{b}$, so $y = z + \frac{a}{b}$

So $\boxed{y(t) = (y_0 - \frac{a}{b})e^{-bt} + \frac{a}{b}}$ is the solution for (*).

Let's summarise:

Step 1: Write DE (including IV) in terms of deviation of y from steady state value.

Step 2: Solve resulting DE using formula for ~~y'=ky~~ $z' = kz$.

Step 3: Convert solution back in terms of y.

<u>Example</u>

Let $C(t)$ = level of atmospheric CO_2

P = pollution (constant)

M = ~~amount~~ mass of living plants

Assume plants absorb CO_2 at a rate proportional to their mass

$$\frac{dC}{dt} = P - MC$$

Say $P = 100$, $M = 25$, $C(0) = 16$.

Let's ~~sl~~ solve this! Say $P=100$
$M=25$

Step 1: $\frac{dC}{dt} = -25(C-4)$

~~Ne~~ Let $z(t) = C(t) - 4$.

Then $\frac{dC}{dt} = \frac{dz}{dt} = -25z$, $z(0) = C(0) - 4 = 12$

Step 2: So $z(t) = 12e^{-25t}$

Step 3: $\boxed{y(t) = z(t) + 4 = 12e^{-25t} + 4}$