Qualitative Analysis of differential equations II
State-Space Diagrams

Math 102 Section 102
Mingfeng Qiu

Nov. 7, 2018

Due due due due due...

- Nov 7 (Today): Pre-lecture 10.2
- Nov 8 (Thursday): Assignment 9
- Nov 9 (Friday): OSH 5

Assignments due: 9:00 pm

Annoucements

- Guest lecturer on Friday (Daniel Di Benedetto)
- How to interpret your midterm score (Link) (Sections 1-2)
- Advice: read your textbook, and do practice exercises

Last time

- A slope field is a geometrical representation of slopes of tangent lines to solution curves of a DE
- One can qualitatively sketch how the solution curves behave with the help of the slope field.

Today: learning goals

- Explain the concept of steady states and their stability.
- Use the slope field to identify steady states and their stability.
- Sketch the state-space diagram of a DE.
- Describe qualitative behavior of the solutions of a DE using the state-space diagram.
    - A state-space diagram is basically the same thing as a slope field, just a biiiiiit more abstract.

Steady state

- A DE $\frac{dy}{dt} = f(y)$ describes a system. The dependent variable y describes the state of the system.

Definition (Steady state)

A steady state is a state in which a system is not changing.

The logistic equation: steady states

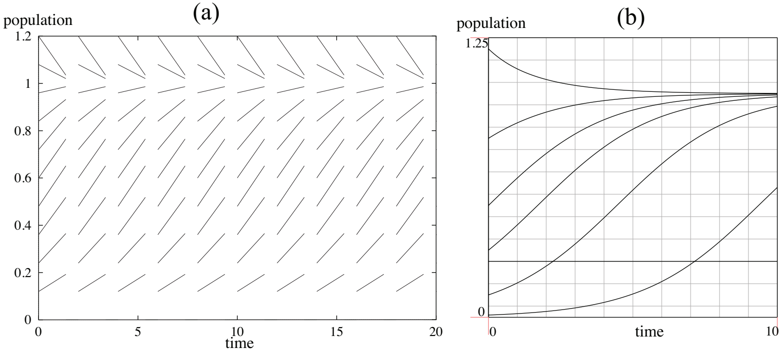
Example

What is the steady state(s) of the logistic equation?

$$\frac{dy}{d\tau} = y(1-y)$$

- Set
$$\frac{dy}{d\tau} = 0$$
- We get $y = 0,\ 1$

The logistic equation: slope field



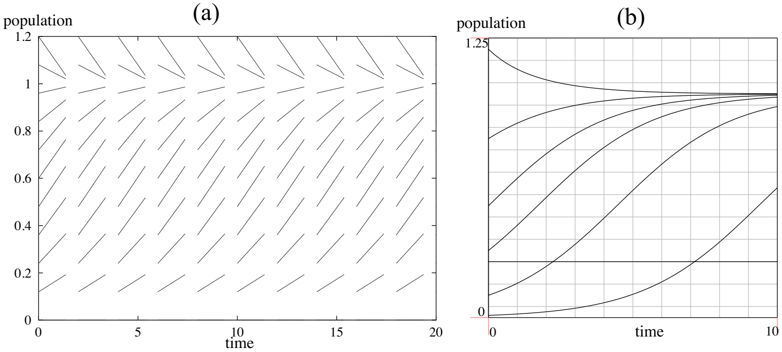
- Can you tell the steady states of the logistic equation from the slope field?
- Do the solutions have an inflection point?
- Yes! Look at the slopes at a fixed point in time.

Stability of steady states

Definition (Stability of steady states)

A steady state S is said to be stable if states initially close enough to S will get even closer in time. S is saild to be unstable if states initially in a neighbourbood of S move away in time.

The logistic equation: slope field



Are the steady states stable?

- $y = 0$ is unstable.
- $y = 1$ is stable.

Observing the slope field from a different angle

Document camera

State-space diagram

Definition (State-space diagmra)

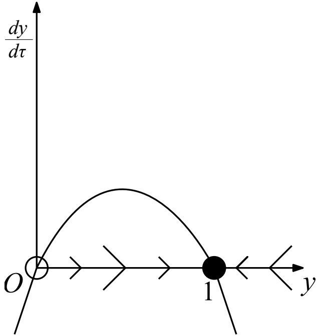
A state-space diagram (or phase line diagram) consists of

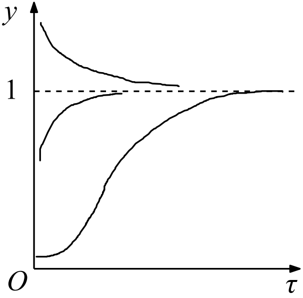
1. A line representing the dependent variable y;
2. Circles (empty and solid) representing steady states;
3. Arrows indicating flow direction along the line.

Sketch the state-space diagram from the DE

$$\frac{dy}{d\tau} = y\,(1 - y)$$

Document camera



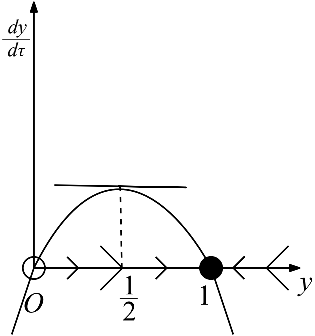


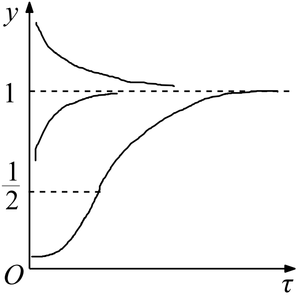
Inflection points in solutions

$$\frac{dy}{dt} = f(y)$$

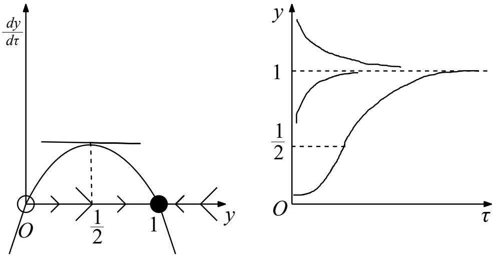
Where are inflection points in solutions?

- At extrema of $f(y)$.





Interpreting results



- The population size tends to $y = 1$ $(N = K)$ regardless what size it starts with.
- If the initial population is small $0 < y(0) < 1/2$ $(0 < N(0) < K/2)$, then the rate of population increase is slow initially, and gradually picks up speed, before dropping down to be slow again when approaching the steady population size.

Summary

- Qualitative analysis (slope fields and state-space diagrams) allows us to understand the behaviour of complex systems when the DEs cannot be easily solved

Related Exam Problems

1. Given the differential equation and initial condition

$$\frac{dy}{dt} = y^2(y - a), \quad y(0) = 2a$$

determine

$$\lim_{t \to \infty} y(t).$$