Qualitative Analysis of differential equations I
Slope Fields

Math 102 Section 102
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Annoucement

- OSH remarking form: put your group name/number (if working in a group) on the form!

Today: learning goals

1. Derive DEs and scale them (cont'd from last Friday)
   - Law of mass action
   - Logistic growth
2. Use slope fields to sketch solutions of DEs

Law of mass action

- Chemical reactions happen because mixed molecules collide into each other. The concentration of any reactant increases $\rightarrow$ The possibility of collision becomes higher $\rightarrow$ The reaction is faster.
- Law of mass action: rate of reaction $= k\times$ (concentration of reactant 1) $\times$ (concentration of reactant 2) $\times \cdots$.

Law of mass action

In a chemical reactor with a constant volume, substance A is added at a constant rate of change of concentration I. Three molecules of A react to form a product. Derive a differential equation to describe the concentration of A in the reactor $a(t)$. The reaction rate coefficient is k.

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- Nonlinearity arises due to the nature of chemical reactions!

Your turn...

Q1. Suppose one molecule of A and two molecules of B combine in a chemical reaction following the law of mass action. If the concentrations of A starts out as half that of B, give an equation for $\frac{da}{dt}$, where $a(t)$ is the concentration of molecule A.

A. $\frac{da}{dt} = -2ka$

B. $\frac{da}{dt} = -ka^3$

C. $\frac{da}{dt} = -2ka^2$

D. $\frac{da}{dt} = -4ka^3$

E. None of the above

Logistic growth

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- Uncontrolled population growth

$$\frac{dN}{dt} = rN$$

- This is not realistic, since there is only limited amount of resources. A common modification to the above model is logistic growth

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

- The nonlinear term $-r\frac{N^2}{K}$ controls population growth when N gets big.

Scaling (Nondimensionalization)

- Notice the logistic growth equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

- It has two parameters $r,\ K$. Both have units. Are they both important mathematically?

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- One equation rules all!

Summary

- Differential equations often describe a scientific process (physics, chemistry, biology).
  - You should be able to write down a DE given a description
  - Key idea: write down a balance relationship, and put it in a mathematical formula
- Using scaling (Nondimensionalization), one can obtain simplified equations which represent the behaviour of a family of equations.

Sketch solutions to differential equations using slope fields

Slope fields

Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y)$$

- If graphing the general solutions:a family of curves on the $y - t$ plane.
- The tangent line to any curve at a point (t, y) has slope equal to$f(y)$ since $\frac{dy}{dt} = f(y)$.
- Sometimes, the equations are hard to solve. But...
- If we know how the slopes look like (called slope fields) in the whole $y - t$ plane, then we would have a very good idea on what the solutions look like.
- Keep in mind that the slope field is continuous—at every point, there is a slope value.

Slope fields

Example:

$$\frac{dy}{dt} = 2y$$

- Idea: $\frac{dy}{dt} = 2y$ is the slope of the tangent line to the solution $y(t)$.

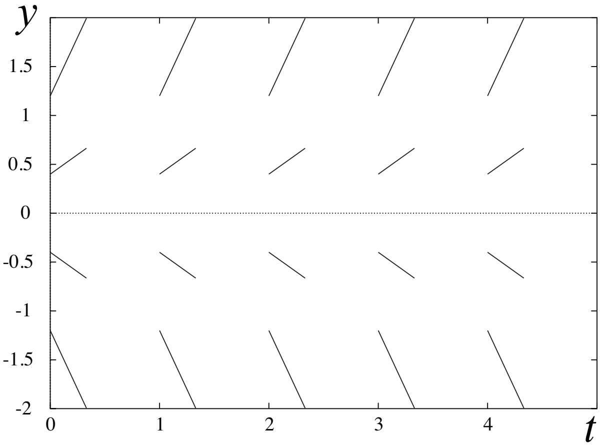
Slope fields

Step 1: calculate a few representative slopes at different y values.

y	$f(y) = 2y$	slope of tangent line	behaviour of y	direction of arrow
-2	-4	-ve	decreasing	$\searrow$
-1	-2	-ve	decreasing	$\searrow$
0	0	0	no change in y	$\rightarrow$
1	2	+ve	increasing	$\nearrow$
2	4	+ve	increasing	$\nearrow$

Slope fields

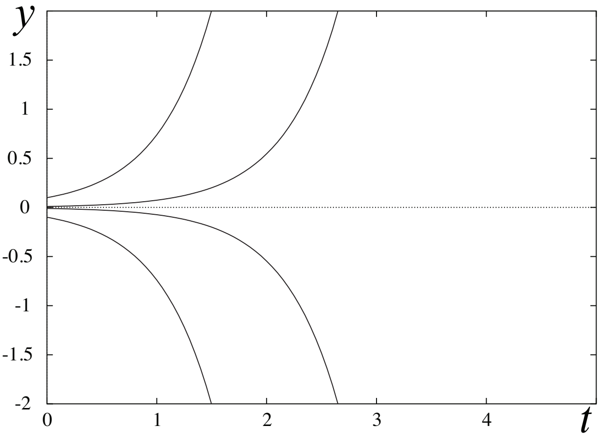
Step 2: On the $y - t$ plane, draw short segments of tangent lines at the same t value. Replicate for other t values.



(a)

Slope fields

Step 3: Start from a few different initial points, sketch solution curves following the tangent line segments field.



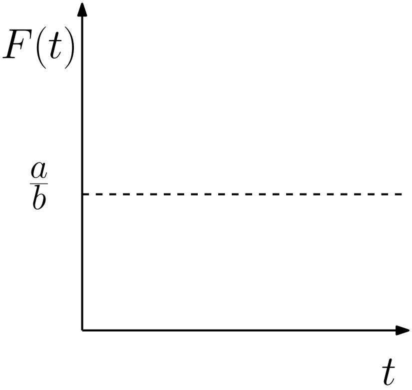
(b)

Your turn...

Sketch the slope field for

$$\frac{dF}{dt} = a - bF$$

on the following set of axes. Then sketch some solution curves.

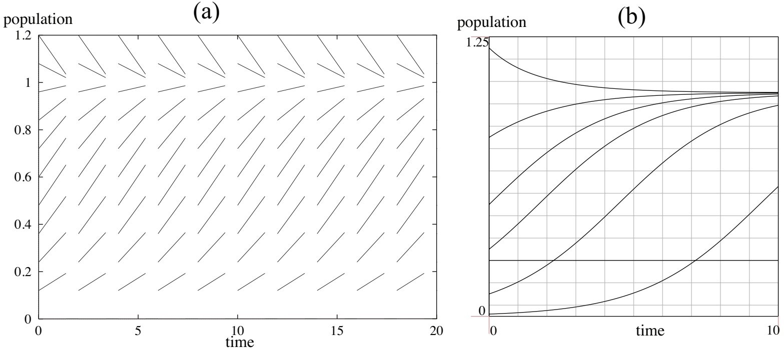


The logistic equation

$$\frac{dN}{d\tau} = y\,(1 - y)$$

- What do the solutions to the logistic equation look like?

The logistic equation



- Do the solutions have an inflection point?

Summary

- The general solutions to a DE can be graphed as a family of curves.
- A slope field is a geometrical representation of slopes of tangent lines to these curves.
- The slope field can sometimes be obtained without solving the DE.
- One can qualitatively sketch how the solution curves look like with the help of the slope field.

Answers

1. D